Homework # 3 Solutions

1. The change in the electric potential energy of the system when the charge in the lower left hand corner of the rectangle is brought to that position from infinity (assuming the other charges have fixed positions) is obtained by computing the potential energy of that charge with each of the other three charges as folllows:

$$U = k_e q^2 \left(\frac{1}{L} + \frac{1}{W} + \frac{1}{\sqrt{L^2 + W^2}} \right).$$

2. (a) The electric potential energy of the three charge configuration is

$$U = k_e \left(\frac{q_1 q_2}{|r_{12}|} + \frac{q_1 q_3}{|r_{13}|} + \frac{q_2 q_3}{|r_{23}|} \right),$$

which here takes the form

$$\begin{aligned} U &= k_e \left(\frac{(20\,\mathrm{nC})(10\,\mathrm{nC})}{.04\mathrm{m}} - \frac{(20\,\mathrm{nC})(10\,\mathrm{nC})}{.04\mathrm{m}} - \frac{(20\,\mathrm{nC})(20\,\mathrm{nC})}{.08\mathrm{m}} \right) \\ &= -4.5 \times 10^{-5}\,\mathrm{J}. \end{aligned}$$

(b) The potential energy of the fourth particle is

$$U = qV,$$

where V is the potential due to the other three charges at the fourth particle's location:

$$V = k_e \left(\frac{20\,\mathrm{nC}}{\sqrt{.03^2 + .04^2}\mathrm{m}} - \frac{20\,\mathrm{nC}}{\sqrt{.03^2 + .04^2}\mathrm{m}} + \frac{10\,\mathrm{nC}}{.03\mathrm{m}} \right) = 2997\,\mathrm{J/C}.$$

Note that we have taken the reference value for the potential at infinity. The potential energy is then

$$U = qV = (40\,\mathrm{nC})(2997\mathrm{J/C}) = 1.2 \times 10^{-4}\,\mathrm{J}.$$

We can now use conservation of energy, noting that the particle's initial kinetic energy is zero and its final potential energy is zero:

$$U_i = \frac{1}{2} m v^2 = K_f.$$

Solving for v, we find:

$$v = \sqrt{\frac{2U_i}{m}} = \sqrt{\frac{2(1.2 \times 10^{-4}\,\mathrm{J})}{2.00 \times 10^{-13}\,\mathrm{kg}}} = 34600\,\mathrm{m/s}.$$

3. We are given $V(x, y, z) = nx - px^2y + 2yz^2$, where n and p are integers. From the definition

$$\vec{E} = -\nabla V$$

the components of the electric field are given by

$$\begin{aligned} E_x &= -\frac{\partial V}{\partial x} = -n + 2pxy \\ E_y &= -\frac{\partial V}{\partial y} = px^2 - 2z^2 \\ E_z &= -\frac{\partial V}{\partial z} = -4yz. \end{aligned}$$

The magnitude of the field is

$$|\vec{E}| = \sqrt{E_x^2 + E_y^2 + E_z^2} = \sqrt{(-n + 2pxy)^2 + (px^2 - 2z^2) + (4yz)^2}.$$

4. (a) The charge distribution is $\lambda = \alpha x$, with α a positive quantity. The units of α are C/m^2.

(b) The origin of coordinates is placed at the left-hand end of the rod, as in the diagram. To get the potential at point A, which is at location $-d$ on the x axis, we must integrate over the length of the rod, from $x' = 0$ to $x' = L$:

$$k_e \alpha \int_{x'=0}^{x'=L} \frac{dx' x'}{x' + d}.$$

To do the integral, make the substitution $z = x' + d$. Then $dz = dx'$ and $x' = z - d$. The integrand becomes $(z - d)/z = (1 - z/d)$, and the limits of integration are from $z = d$ to $z = L + d$:

$$k_e \alpha \int_{z=d}^{z=L+d} (dz(1 - d/z)) = k_e \alpha (z - d\ln(z))|_{z=d}^{z=L+d},$$

which simplifies to

$$k_e \alpha (L + d - d - (d\ln(L + d) - d\ln(d))) = k_e \alpha (L - d\ln(1 + L/d)).$$

5. Each element of charge dq is an equal distance R from point O, where R is related to the length L of the rod by

$$R = \frac{L}{\pi}.$$

The potential at O is then given by

$$V = k_e \int \frac{dq}{r} = k_e \frac{1}{R} \int dq = \frac{k_e Q}{R},$$

where Q is the total charge on the rod.

6. (a) Inside the spherical conductor, the electric field is zero. The potential is constant inside the conductor (taking the reference value of the potential at ∞):

$$V = \frac{k_e Q}{R},$$

where R is the radius of the conductor and Q is its charge.

(b) Outside the conductor ($r > R$), the electric field is

$$E = \frac{k_e Q}{r^2}.$$

It points in the outward radial direction if $Q > 0$, and in the inward radial direction if $Q < 0$. The potential outside the conductor is

$$V = \frac{k_e Q}{r}.$$

(c) At the surface of the conductor ($r = R$), the electric field is

$$E = \frac{k_e}{Q} R^2.$$

Again, it points in the outward radial direction if $Q > 0$, and in the inward radial direction if $Q < 0$. The potential is

$$V = \frac{k_e Q}{R}.$$

7. (a) The capacitance is given by

$$C = \frac{Q}{\Delta V}.$$

(b) Given the capacitance and the new charge Q', the new potential difference $\Delta V'$ can be obtained by

$$\Delta V' = \frac{Q'}{C}.$$

8. We are given two conducting spheres of diameters d_1 and d_2 (radii $r_1 = d_1/2$ and $r_2 = d_2/2$), connected by a long wire (length $L \gg d_{1,2}$) and with a total charge Q.

(a) Denote the charges on spheres 1 and 2 as q_1 and q_2. The spheres are at the same potential:

$$\frac{k_e q_1}{r_1} = \frac{k_e q_2}{r_2},$$

such that

$$\frac{q_1}{r_1} = \frac{Q - q_1}{r_2}.$$

Solving for q_1 yields

$$q_1 = \frac{Q r_1}{r_1 + r_2}, \quad q_2 = Q - q_1 = \frac{Q r_2}{r_1 + r_2}.$$

(b) The system of spheres is an equipotential with $V_1 = V_2 = V$ (think of them as capacitors in parallel), where

$$V = \frac{k_e q_1}{r_1} = \frac{k_e q_2}{r_2} = \frac{k_e Q}{r_1 + r_2}.$$

9. Denote the area of the capacitor plates as A and the spacing as d. Then,

$$Q = C\Delta V = \frac{\epsilon_0 A}{d}\Delta V.$$

Since the surface charge density σ is

$$\sigma = \frac{Q}{A} = \frac{\epsilon_0}{d}\Delta V,$$

the spacing d is given by

$$d = \frac{\epsilon_0 \Delta V}{\sigma}.$$

10. First, combine the series combinations in the middle and lower branches. The capacitance of the middle branch is

$$C_{\text{middle}} = \left(\frac{1}{C} + \frac{1}{C}\right)^{-1} = \frac{C}{2},$$

and the capacitance of the lower branch is

$$C_{\text{lower}} = \left(\frac{1}{C} + \frac{1}{C} + \frac{1}{C}\right)^{-1} = \frac{C}{3}.$$

The two capacitors C_{middle} and C_{lower} are now in parallel with the capacitor C on the upper branch, such that the total effective capacitance is

$$C_{\text{eff}} = C + \frac{C}{2} + \frac{C}{3} = 1.83C.$$