Scalars of the Universe

10^3 stars + clouds of gas & dust

- 10^3 kpc
- 8 kpc
- 0.3 kpc

\[ V = 10 \text{ cm} \]

Hubble radius: 14 G years \( \approx 45,000 \text{ Mpc} \)

Satellite galaxies: LMC (55 kpc), Fornax ...

Local cluster: Andromeda (2 Mpc), M33 ...

\[ r \approx 10 \text{ Mpc} \]

Local supercluster: Virgo (20 Mpc), M81, Coma, Ursa Major ...

\[ r \approx 100 \text{ Mpc} \quad \text{(nearest quasars)} \]

\[ L \approx 10^{45} \sim 10^{48} \text{ erg/sec} \]

Cepheids \(< 5 \text{ Mpc} \)

- Redshift
- Hubble flow
- Nucleosynthesis
- Microwave background

\[ \Lambda \text{obs} > \Lambda \text{rest} \]

\[ \frac{\Lambda \text{obs}}{\Lambda \text{rest}} = \frac{c + u}{c} \quad \text{for small } u \]

\[ \sqrt{1 - \frac{u^2}{c^2}} = \frac{c + u}{c} = 1 + z \]

\[ z = \frac{u}{c} \quad \text{for small } u \]

Doppler shift
clock slows down
Table 2.1 Approximate sizes and masses in the universe
(1 parsec = 1 pc = 3.09 × 10^{16} m = 3.26 lightyears)

<table>
<thead>
<tr>
<th></th>
<th>Radius</th>
<th>Mass</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sun</td>
<td>7 × 10^8 m</td>
<td>2 × 10^{30} kg = M_☉</td>
</tr>
<tr>
<td>Galaxy</td>
<td>15 kpc</td>
<td>10^1 M_☉</td>
</tr>
<tr>
<td>Cluster</td>
<td>5 Mpc</td>
<td>10^4 M_☉</td>
</tr>
<tr>
<td>Supercluster</td>
<td>50 Mpc</td>
<td>10^5 M_☉</td>
</tr>
<tr>
<td>Universe</td>
<td>4500 Mpc</td>
<td>10^{23} M_☉</td>
</tr>
</tbody>
</table>

In fact astronomers use a logarithmic scale of luminosity, called **magnitude**, running (perversely) from small values of magnitude for the brightest stars to large values for the faintest. The defining relation between the apparent magnitude \( m(z) \) at redshift \( z \), the so-called absolute magnitude \( M \) (equal to the value that \( m \) would have at \( D_L = 10 \) pc) and the distance \( D_L \) in Mpc, is given by the **distance modulus**

\[
m(z) - M = 5 \log_{10} D_L(z) + 25
\]  
(2.3)

In the Hubble diagram, \((m - M)\) or \(\log_{10} D_L\) is plotted against \(\log_{10} z\).

A modern version of the Hubble plot at small redshifts is shown in Fig. 2.3, for events of \( z < 0.1 \). The various sources in this plot include, for example, Cepheid variable stars for \( z < 0.01 \), and Type Ia and Type II supernovae for higher redshifts. Cepheid variables can be used as 'standard candles', since they vary in luminosity due to oscillations of the envelope, the period \( \tau \) being determined by the time for sound waves to cross the stellar material—\( \tau \propto T^{0.8} \).

Supernovae, discussed in Chapter 7, signal the death throes of stars in the final stages of evolution, and when they occur, their light output for a time—typically weeks or even months—can completely dominate that from the local galaxy. So, in principle, they are useful for probing out to large distances and redshifts, or equivalently, back to earlier times. The absolute distance scale to the thirty or so nearest spiral galaxies where a few Type Ia or Type II supernovae have occurred, has been established by observations on Cepheid variables, and this provides a means of calibrating supernova luminosity.
Universe as the surface of a balloon or FLRW in 2D space

\[
\frac{x_1^2 + x_2^2 + x_3^2}{k} = \frac{R^2}{k}
\]

\(k=1\) is balloon

\[
ds^2 = (d\theta d\phi) g \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}
\]

\[
g = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2 \theta \end{pmatrix}
\]

\[
d\Omega^2 = d\Omega^2 + R^2 d\theta^2 + R^2 \sin^2 \theta d\phi^2 = R^2 d\Omega
\]

Volume \(V\) and curvature \(K\)

\[
V = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sqrt{\det g} = 2\pi R^2 \int_0^\pi \sin \theta d\theta = 4\pi R^2
\]

(area of balloon)

\[
K = \lim_{\delta \to 0} \frac{(2\pi \theta - C)}{\delta^3}
\]

\[
2\pi \delta = 2\pi R \sin \frac{\delta}{R} = 2\pi R \left( \frac{\delta}{R} - \frac{1}{3!} \frac{\delta^3}{R^3} \right)
\]

\[
K = \frac{k}{R^2} \quad \left( \frac{1}{R^2} \text{ for balloon} \right)
\]
Problem: Draw the universe*

*Assume 2D like a balloon

**Closed surface**
\[ C < 2\pi a \]
\[ x_1^2 + x_2^2 + x_3 = R^2 \]
(1D circle)

**Open surface**
\[ C > 2\pi a \]
\[ x_1^2 + x_2^2 = x_3^2 = -R^2 \]
(1D hyperbola)

**Flat surface**
\[ C = 2\pi a \]
(1D line)

- no curvature
- \( R \) not defined
- 1D flat
FLRW metric of the real Universe
balloon in 3D, not 2D
space + time
cosmological principle: Universe is homogeneous & isotropic

$$ds^2 = cdt^2 - dl^2$$

$$dl^2 = R(t)^2 \left[ \frac{dr^2}{1 - k \ell^2} + \ell^2 d\Omega^2 \right]$$

$$\ell^2 = x_1^2 + x_2^2 + x_3^2 + \frac{x_4^2}{k} = \frac{R^2(t)}{k}$$

$$x_1 = R(t) \ell \sin \theta \cos \varphi$$

$$x_2 = R(t) \ell \sin \theta \sin \varphi$$

$$x_3 = R(t) \ell \cos \theta$$

$$x_4 = \sqrt{1 - k \ell^2} \cdot R$$

scale factor

$$\text{co-moving coordinate} \quad (\text{see later})$$

time independent!

radial coordinate is $R(t) \ell$

to express the fact

that the time dependence is the same for all $\ell$. Different

time dependence at $\ell_1, \ell_2$ violates cosmological

principle → would create anisotropy.
FLRW revisited

Using coordinates \((cdt, dz, d\Theta, d\Phi)\)

\[
g = \begin{bmatrix}
1 & -\frac{R^2}{1-kr^2} & -R^2r^2 & -R^2r^2\sin^2\Theta \\
-\frac{R^2}{1-kr^2} & -R^2r^2 & R^2r^2 & R^2r^2\sin^2\Theta \\
-R^2r^2 & R^2r^2 & -r^2 & 0 \\
-R^2r^2\sin^2\Theta & R^2r^2\sin^2\Theta & 0 & -r^2
\end{bmatrix}
\]

- **Curvature at** \(\Theta = \Phi = 0\) **from Riemann**

\[
K = \frac{1}{2g_{11}g_{22}} \left\{ -\frac{\partial^2 g_{11}}{\partial z^2} + \frac{1}{2g_{11}} \left[ \frac{\partial g_{11}}{\partial x_1} \frac{\partial g_{22}}{\partial x_2} + \left( \frac{\partial g_{11}}{\partial x_2} \right)^2 \right] + (x^2) \right\}
\]

\[
= -\frac{R^2}{R}
\]
- \( l \) is the co-moving coordinate

\[
l = \int dl = \int_0^\tau \frac{R(t)}{\sqrt{1-k \gamma^2}} \, dt
\]

The present distance depends on how the universe evolved since the light was emitted.

**Flat**

\[
l = R(t) \gamma
\]

- \( k = 0 \)
  - \( l = 0 \) and \( l = 2\pi R \)
  - \( \tau = \sin \frac{l}{R} \)

**Long**

\[
l = R \sin^{-1} \gamma
\]

**Cyclic universe**

\[
l = R \exp^{-1} \gamma
\]
Hubble flow: calculate recession velocity $\dot{r}$

$$V = \dot{r} = R \int_{0}^{\tau} \frac{dv}{\sqrt{1-kv^2}} = \frac{\dot{R}(t)}{R(t)} \frac{r}{R(t)}$$

$$\dot{r} = H(t) r \quad H(t) = \frac{\dot{R}(t)}{R(t)}$$
Einstein: curvature proportional to mass

\[ K \propto g \]

or, from dimensional analysis:

\[ K = -\frac{\ddot{R}}{R} = \alpha G^m c^m \rho + \text{cte} \]

Curvature not related to mass: violates Mach

\[ [K] = \frac{1}{T^2} \quad [G] = \frac{L^3}{MT^2} \quad [c] = \frac{L}{T} \quad [\rho] = \frac{M}{L^3} \]

\[ K = -\frac{\ddot{R}}{R} = \alpha G \rho \]

What is the proportionality constant \( \alpha \)?

Answer: Newtonian limit
Force on galaxy of mass $m$ only depends on $\rho(t)$ inside the volume of radius $R(t)$.

Newton's law

$$m \ddot{R} = -G m \frac{4\pi}{3} \rho (2R)^3 (2R)^2$$

$$-\frac{\ddot{R}}{R} = \frac{4\pi}{3} G \rho$$

Friedman I

Therefore, $\alpha = \frac{3\pi}{2}$

Energy of $m$:

Multiply both sides with $\dot{R}$ and use mass conservation:

$$M = \rho R^3 = \rho_0 R_0^3$$

$$m \ddot{R} = -G \frac{mM}{R^2} \dot{R}$$

Integrate

$$\frac{1}{2} m \dot{R}^2 - \frac{mM}{R^2} = 0 \rightarrow \text{total energy is zero}$$
Previous derivation in flat universe $k=0$

For $k \neq 0$ particles and photons move on geodesics (because the scatter of gravitational fields)

$$\frac{1}{2} m \dot{R}^2 - \frac{mM}{R} = -k \frac{1}{2} mc^2 \quad \text{(GR result)}$$

$$\frac{\dot{R}^2}{k} = \frac{8\pi}{3} j - \frac{k c^2}{R^2} \quad \text{Friedman II}$$

Introduce cosmological constant (long range repulsive force proportional to $R$)

$$\dot{R} = -\frac{\dot{R}}{R} = \Lambda G \rho (t) - \frac{\Lambda}{3}$$

Therefore the Friedman equations become

$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3} G \rho + \frac{\Lambda}{3}$$

$$\frac{\dot{R}^2}{R} = \frac{8\pi}{3} G \rho - \frac{k c^2}{R^2} + \frac{\Lambda}{3} \frac{R^2}{R}$$
Hubble flow revisited

we only measure the Universe now \((t=0)\)

\[ R(t) = R_0 + \dot{R}(t-t_0) + \frac{1}{2} \ddot{R}_0 (t-t_0)^2 \]

\[ H_0 = \frac{\dot{R}_0}{R_0} \quad q_0 = -\frac{1}{H_0^2} \frac{\ddot{R}_0}{R_0} \]

\[ R(t) = R_0 \left[ 1 + H_0 (t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2 \right] \]

Definition of the observables \(H_0\) (Hubble constant) and \(q_0\) (acceleration parameter)

\[ \Lambda = 4\pi G \rho_0 - 3q_0 H_0^2 \]

\[ k = \frac{R_0^2}{c^2} \left[ 4\pi G \rho_0 - H_0^2 (q_0+1) \right] \]

For flat Universe with no cosmological constant \(\Lambda = 0\); \(k = 0\)

\[ \rho_0 = \frac{3H_0^2}{8\pi G} \]

\[ q_0 = \frac{1}{2} \]
Possible universes: $\Lambda = \Omega_0$

$k = -1$ open

$$\left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G}{3} \rho + \frac{c^2}{R^2} \rightarrow \frac{c^2}{R^2} \quad (\rho \sim \frac{1}{R^3})$$

$\dot{R} = c \quad R = c t$

expands at constant rate

$k = 0$ flat

$$\left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G}{3} \rho \rightarrow \frac{1}{R^3} \quad (\rho \sim \frac{1}{R^3})$$

$R \propto R^{-\frac{1}{2}} \quad R \propto R^{\frac{1}{2}}$

expands, rate slows down

$k = -1$ cyclic

$$\left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G}{3} \rho - \frac{c^2}{R^2} \quad \dot{R} = 0 \text{ for } R_{\text{max}}$$

R

$R_{\text{max}}$

$M = \frac{4}{3} \pi R_{\text{max}}^3$