How do we interpret the gravitational redshift?

Observers at emitter and receiver measure time differently.

Gravitational time dilation

\[ \omega \sim \frac{1}{d\tau_1} \]

\[ \frac{d\tau_1 - d\tau_2}{d\tau_2} = \frac{\Phi_1 - \Phi_2}{c^2} \]

\[ \Rightarrow d\tau_1 = \left( 1 + \frac{\Delta \Phi}{c^2} \right) d\tau_2 \]

\[ = \left( 1 + \frac{\Delta \Phi}{c^2} \right) \tau \text{ time dilation} \]

For static gravitational field, can integrate to get

\[ \tau_1 = \left( 1 + \frac{\Phi_1 - \Phi_2}{c^2} \right) \tau \]
Note: ① Time dilation even though there is no relative motion.

② Unlike time dilation in SR (i.e., each see other's clock run slower).
Here:
lower clock see upper clock run fast
upper clock see lower clock run slow

③ Time dilation by gravity has been tested.

After correcting for SR effect, clock in airplane still runs fast.
Also: applications in GPS (homework)
More direct derivation of time dilation

Drop a clock from a building

\[ t_{ff} = \gamma_1 \tau_1 \]
\[ t_{2,ff} = \gamma_2 \tau_2 \]

Go to free-falling frame (inertial frame)

\[ \Rightarrow \text{SR applies} \]

We are interested in comparing \( dt_{ff} \) and \( d\tau_2 \)

where \( dt_{ff} = d\tau_{2,ff} \)

\[
\frac{dt_1}{d\tau_2} = \sqrt{\frac{1 - \frac{u_1^2}{c^2}}{1 - \frac{u_2^2}{c^2}}} \approx 1 - \frac{1}{2} \left( \frac{u_1^2 - u_2^2}{c^2} \right) + \ldots
\]

\[ = 1 + \frac{\phi_1 - \phi_2}{c^2} \]

Conservation of energy

\[ \frac{1}{2} m u_i^2 + \phi_i = \text{constant} \]
Bending of light

First, qualitative:

Free-falling frame = inertial frame

light travels horizontally
with speed = c

Now according to our frame, elevator
is dropping so

light bends like
a parabola
under influence
of gravity

We want to calculate the angle 8 light
is bent in the presence of a mass M
Let us get the answer first by cheating.

\[ \Delta p_y = \int F_y \, dt = \int F \cos \theta \, dt \]

\[ = \int \frac{G M m}{r^2} \cos \theta \, dt \]

\[ = \int \frac{G M m}{r^2} \cos \theta \left( \frac{dx}{c} \right) \]

\[ = \int_{-\infty}^{\infty} \frac{G M m}{c} \frac{\cos \theta \, dx}{x^2 + r_{\text{min}}^2} \]

\[ = \frac{G M m}{c} \int_{-\infty}^{\infty} \frac{\cos \theta \, dx}{x^2 + r_{\text{min}}^2} \]

\[ = \frac{G M m}{r_{\text{min}} c} \int_{-T/2}^{T/2} \cos \theta \, d\theta \quad \text{where} \quad dV = r_{\text{min}} \omega \, r_{\text{min}}^2 \, d\theta \]

\[ = \frac{2 G M m}{r_{\text{min}} c} \]
\[ \frac{\Delta p_y}{p} = \frac{2GMm}{r_{\text{min}}c} \frac{1}{E/c} = \frac{2GM}{r_{\text{min}}c^2} \]

Other than the fact that this is not the full story for GR, we got this result just because we were lucky.

Problems with this method:

1. Massless particle only makes sense in relativity, do not expect \( \frac{GMm}{r^2} \) to work.

2. Derivation does not take into account gravitational time dilation, which is crucial in the reasoning behind bending of light.
Bending of light (from EP)

Speed of light \[ \frac{dr}{dT} = c \] universal constant

According to an observer at a different position, speed appears to be different because gravitational potential is different.

Let's choose the observer at infinity \( t = \tau (\infty) \), \( \Phi (\infty) = 0 \)

\[ dr = (1 + \frac{\Phi(r)}{c^2}) \, d\tau \]

Apparent speed of light

\[ C(r) = \frac{dr}{dT} = (1 + \frac{\Phi(r)}{c^2}) \frac{dr}{dT} = (1 + \frac{\Phi(r)}{c^2})c \]

\[ = \frac{c}{n(r)} \]

where \( n(r) \) can be interpreted as an index of refraction

\[ n(r) = (1 + \frac{\Phi(r)}{c})^{-1} \approx 1 - \frac{\Phi(r)}{c^2} \]
Important to Note

- \( c = \frac{dr}{dt} \) is still a universal constant rather it is the clock which runs differently so it appears to change according to a fixed clock

- A dramatic example: Black hole

Infinite time dilation \( \sim \) appears to an observer at \( \infty \) that it takes an amount of time to leave the black hole even though proper time is finite.
Bending of light (the EP expectation)

\[ g = 0 \]

\[ d \delta \approx \frac{(c_1 - c_2) dt}{dy} \quad \frac{dc(r)}{dy} \frac{dx}{c} \]

\[ = \left( \frac{\partial n^{-1}}{\partial y} \right) dx \]

\[ = - \frac{1}{n^2} \left( \frac{\partial n}{\partial y} \right) dx \]

\[ = - \left( \frac{\partial n}{\partial y} \right) dx \]

\[ n = 1 - \frac{\phi(r)}{c^2} \]

\[ \frac{\partial n}{\partial y} = \frac{\partial n}{\partial r} \frac{\partial r}{\partial y} = - \frac{1}{c^2} \frac{\partial \phi}{\partial r} \frac{y}{r} \]

\[ = \frac{GM}{c^2 r^2} \frac{y}{r} \]

\[ \delta = \int_{-\infty}^{\infty} \frac{GM y}{c^2 r^3} dx = \frac{GM}{c^2} \int_{-\infty}^{\infty} \frac{y}{r^3} dx \]
For small bending, \( y \ll r_{\min} \)

\[
\Rightarrow \delta = \frac{GM}{c^2} \int \frac{r_{\min} \, dx}{(x^2 + r_{\min}^2)^{3/2}}
\]

\[
= \frac{2GM}{c^2 r_{\min}}
\]

For \( r_{\min} = R_\odot \quad M = M_\odot \)

\[
\uparrow \quad \text{sun}
\]

\[
\Rightarrow \delta = 0.875''
\]

Turns out only \( \frac{1}{2} \) of the correct answer for GR.

Experimental verification of such (correct) angle of deflection is one of the greatest triumphs of GR.

Application

Gravitational lensing

Galaxy cluster

\[ \uparrow \]

Massive object acts as a lens

See distinct galaxies that are otherwise out of sight.