Generalized Geometrical Objects in Metric Spaces

James N. Bellinger

March 27, 2013

Abstract

I investigate some questions resulting from a simple definition of a straight line in an arbitrary metric space. The triangle inequality is used as a "line equality," and defines the line. I try to reproduce some of the definitions for simple geometrical objects and look at the differences from the Euclidian.

1 Introduction

A traditional metric space is defined by a set and a mapping ρ from pairs of elements in that set to the reals such that:

1. $\rho(x, y) \geq 0$
2. $\rho(x, y) = \rho(y, x)$
3. $\rho(x, y) = 0 \Longleftrightarrow x \equiv y$
4. $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$

It is desirable, since spacetime is not a traditional metric space, to extend this study into modified metric spaces which do not satisfy the first criterion. First things first, though– traditional metric spaces are easier to understand.

Presumably there are sets for which no possible metric space can have some particular property, and it may be of interest to classify these sets on such a basis, but that must wait until we have a better handle on which properties are likely to be interesting.

Some of the properties the metric space can have are those of being open or closed or neither, of being bounded, of being compact, and so on. Note that different metrics defined over the same space may result in metric spaces with different properties.

1.1 Examples of Metric Spaces

I believe it is important to have some examples available to hand on which one may test various hypotheses and get a feeling for the territory.

1. The simplest metric space is the trivial metric, in which $\rho(x, x) = 0$ and $\rho(x, y) = 1$ when $x \not\equiv y$.
2. Two more simple spaces are R^2 with the metric $\rho(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$: (the 'as the crow flies' metric), and
3. R^2 with the metric $\rho(x, y) =\mid (x_1 - y_1) \mid + \mid (x_2 - y_2) \mid$: (the 'city streets' (or taxi-cab or L_1) metric).
4. R^3 with a hybrid metric $\rho(x, y) = \sqrt{(x_3 - y_3)^2 + (\mid (x_1 - y_1) \mid + \mid (x_2 - y_2) \mid)^2)}$ is interesting. Call it a 'city crow' metric–like a bird constrained to fly between skyscrapers.
5. Another is the distance between two points on the unit sphere. If two points have polar coordinates θ_1, ϕ_1 and θ_2, ϕ_2 then in the 'great circle' metric the distance between them is given by $\sin(\theta_1) \sin(\theta_2) \cos(\phi_1 - \phi_2) - \cos(\theta_1) \cos(\theta_2)$.

6. Consider R^2 with the point $(0,0)$ missing. Let it have coordinates (r,θ), and define the metric such that for points x_1 and x_2, $\rho(x_1,x_2) = \min(\int_{x_1}^{x_2} ds/r)$, where the integral is taken over some path from x_1 to x_2. The result (call it the 'potential well metric') has several cases. If $x_1 = (r_1,\theta)$ and $x_2 = (r_2,\theta)$ (the same θ), then $\rho(x_1,x_2) = |\ln(r_1/r_2)|$. If the radii are the same, then $\rho(x_1,x_2) = |\theta_2 - \theta_1|$. Otherwise

$$\rho(x_1,x_2) = \sqrt{((\theta_2 - \theta_1)^2 + (\ln(r_2/r_1)^2)}$$

7. Consider the familiar space consisting of two copies of R^2, an 'upper' and a 'lower', which are joined along the negative x-axis such that the upper -y joins the lower +y, and the lower -y joins the upper +y. This space may be parameterized by r and θ where θ runs from 0 to 4π. The distance I select is the obvious extension of the 'crow flies' distance: For two points (r_1,θ_1) and (r_2,θ_2), if $|\theta_1 - \theta_2| < \pi$ then the distance is familiar: $\sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$, but if $|\theta_1 - \theta_2| > \pi$ then the shortest distance between the two points is to $(0,0)$ and back, and so is $r_1 + r_2$.

8. Consider the metric space (call it Dis4) over 4 points $\{A,B,C,D\}$, with distances between them defined by:

$$\begin{array}{ccccc} & A & B & C & D \\ A & 0 & 2/5 & 1/3 & 1 \\ B & 2/5 & 0 & 1/4 & 3/5 \\ C & 1/3 & 1/4 & 0 & 2/3 \\ D & 1 & 3/5 & 2/3 & 0 \end{array} \tag{1}$$

9. For a nice pathological case, consider the real interval $[0,2]$, with $\rho(a,b) =| a - b |$ if a and b are rational, and $= 1$ if either of a or b is not rational. Call this Rat2.

2 Line Segments and Lines

In a familiar line segment, the distance from an end-point to a point in the middle plus the distance from that point to the other end-point is equal to the distance between the end-points. This seems to extend very naturally to traditional metric spaces, namely:

$$S_{a,b} \equiv \{x \mid \rho(a,x) + \rho(x,b) = \rho(a,b)\} \tag{2}$$

It is simple to extend this definition to an entire line defined by two points:

$$L_{a,b} \equiv \{x \mid \rho(a,x) + \rho(x,b) + \rho(a,b) = 2 * \max(\rho(a,b), \rho(a,x), \rho(x,b))\} \tag{3}$$

2.1 Specific Examples of $S_{a,b}$

Obviously if $a \equiv b$ then $S_{a,a} = \{a\}$ and $L_{a,a}$ is the entire set. It is also immediate that the trivial metric does not have any interesting line segments ($S_{a,b} = \{a,b\}$).

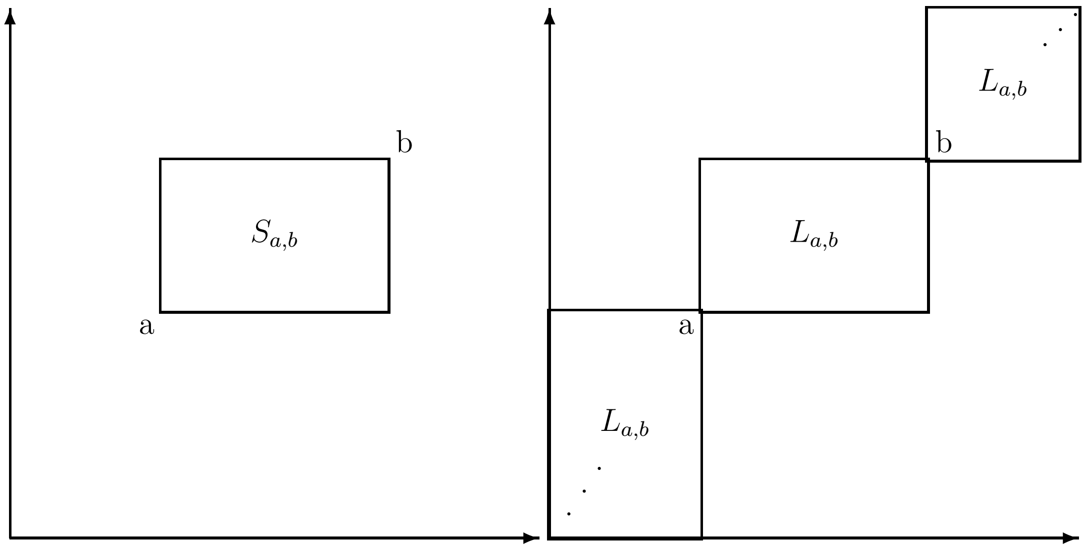

Figure 1: $S_{a,b}$ and $L_{a,b}$ in taxicab metric

The 'as the crow flies' metric behaves just as expected, but in the 'city streets' metric a line segment is, in general, a rectangle with opposite corners at the points a and b, and thus has what one might term 'width' in general. While it is tempting to think of width as the interior of the line segment (and indeed the 'city streets' metric line segments usually do have an interior), in general this isn't possible, as may be seen from the 'city crow' metric. A line in the 'city streets' metric consists of the line segment (usually looks like a rectangle) plus the quadrants tangent to the points a and b and to $S_{a,b}$, as shown in Figure 1.

The 'great circle' metric has line segments which are great circle arcs joining the points a and b, unless these points are exactly opposite each other on the sphere, in which case the entire surface of the sphere is the line segment. A line consists of this arc plus an arc from b to a point exactly opposite from a on the sphere, and from a to a point exactly opposite to b on the sphere.

In the 'potential well' metric space, if the two points have the same θ coordinate, the line segment is a section of radius connecting them. If they have the same radius, the segment is the arc connecting them of the circle they lie on–unless they are exactly opposite in θ, in which case the entire circle is the line segment. If they are neither, then there is a simple equation linking the radius and θ of the arc connecting them: $\theta = ((\theta_2 - \theta_1)/\ln(r_2/r_1))\ln(r/r_1)$. Note again that if the θ's are exactly opposite to each other, there are two arcs, one to each side of the point $(0, 0)$, which are mirror images of each other.

In general, if the metric is defined so that the 'distance' between two points a and b is the minimum of the weighted integral over the path between them (a minimum path), then that path is part of the $S_{a,b}$, by construction.

2.2 $S_{a,b}$ is Sometimes Closed

One obvious question is: Is $S_{a,b}$ closed, open, or neither?

The usual line segment on the 'crow flies' metric is, of course, closed. The segment $S_{0,1}$ on Rat2 consists of all rational numbers in the range $[0, 1]$, and is not closed.

Consider the complement of $S_{a,b}$, and a point Y within that complement. $\rho(a, Y) + \rho(Y, b) = \epsilon + \rho(a, b)$ where $\epsilon > 0$, since Y is not in $S_{a,b}$. Now take a point c within $S_{a,b}$. We have $\rho(a, Y) + \rho(Y, b) = \epsilon + \rho(a, c) + \rho(c, b)$. But $\rho(a, Y) \leq \rho(a, c) + \rho(c, Y)$ and $\rho(Y, b) \leq \rho(c, b) + \rho(c, Y)$, so substituting in the above gives $\rho(a, c) + \rho(c, Y) + \rho(c, b) + \rho(c, Y) \geq \epsilon + \rho(a, c) + \rho(c, b)$, which simplifies to $\rho(c, Y) \geq \epsilon/2$. Thus for any point c in $S_{a,b}$ and any point Y not in it, there exists a minimum separation between them. A circle centered on Y with a radius less than that minimum separation will contain no points of $S_{a,b}$, and thus be a subset of its complement.

If that circle contains only a finite number of points, it is closed, and whether $S_{a,b}$ is open depends on whether it also has only a finite number of points and whether there is an infinite number of points Y.

If, however, the circle is open, then a union of them will also be open. Thus the complement of $S_{a,b}$ is open, and so $S_{a,b}$ will be closed.

2.3 $S_{a,b}$ is bounded

This is easily seen. Let $w = \rho(a, b)$; then defining the circle to be

$$C_{q:r} \equiv \{x \mid \rho(q, x) < r\} \tag{4}$$

clearly each $x \in S_{a,b}$ satisfies $x \in C_{a:w+\epsilon}$, and thus $S_{a,b}$ is bounded. We needn't restrict ourselves to the endpoints, of course. For each $x, y \in S_{a,b}$, we have

$$\rho(x, y) \leq \rho(x, b) + \rho(y, b) \leq 2 * \rho(a, b)$$

and the distance between any two points within $S_{a,b}$ is less than or equal to $2 * w = 2\rho(a, b)$, and $S_{a,b} \subset C_{x,2*w+\epsilon}$.

2.4 Partitionable and Well-ordered

If a line segment has the property that for each point $c \in S_{a,b}$, $S_{a,b} = S_{a,c} + S_{c,b}$ then that line segment is **partitionable** . This is a fairly strong requirement. Segments consisting of only three points are trivially partitionable .

If for each pair of points c and d within a line segment $S_{a,b}$, $\rho(a, c) = \rho(a, d) \iff c = d$ then the line segment is 'well-ordered.' If this is true for all line segments in the space, then the metric space is 'well-ordered.' It can happen that a line segment will have multiple branches, each of which, considered separately, is well-ordered; but the whole segment is not. Line segments of the 'taxi-cab' are generally not well-ordered. If a segment is not partitionable , then for some $x, y \in S_{a,b}$, it must be true that $y \notin S_{a,x} \cup S_{b,x}$. Suppose, however, that for this x, y, $x \in S_{a,y}$. Then $\rho(a, x) + \rho(x, y) = \rho(a, y)$. Adding $\rho(y, b)$ to both sides and using $y \in S_{a,b}$ gives $\rho(a, x) + \rho(x, y) + \rho(y, b) = \rho(a, b) = \rho(a, x) + \rho(x, b)$, since

$x \in S_{a,b}$. From this we find that $\rho(x,y) + \rho(y,b) = \rho(x,b)$, showing that $y \in S_{x,b}$, contrary to our supposition. Hence if $x, y \in S_{a,b}$ and $y \notin S_{a,x} \cup S_{x,b}$, then $x \notin S_{a,y} \cup S_{y,b}$.

'Well-ordered' is an analog of the usual definition of a line segment, which is the set of points $(1-t)x + ty : 0 \le t \le 1$. At first glance partitionable and 'well-ordered' seem to be closely related, and in fact partitionable implies 'well-ordered.' Given $x, y \in S_{a,b}$, we must have $y \in S_{a,x}$ or $y \in S_{x,b}$ if $S_{a,b}$ is partitionable . Without loss of generality assume the first case, $\rho(a,y) + \rho(y,x) = \rho(a,x)$. If $x \neq y$ we have $\rho(y,x) > 0$ and thus $\rho(a,y) \neq \rho(a,x)$. If $x = y$ $\rho(a,y) = \rho(a,x)$ of course. On the other hand, if $\rho(a,y) = \rho(a,x)$, then we have to have $\rho(x,y) = 0$, which implies that $x = y$. So partitionable implies 'well-ordered.'

The metric Dis4 has a non-trivial line segment in $S_{A,D}$, which consists of the entire set. This segment is clearly well-ordered. It is not partitionable . Thus well-ordered does not imply partitionable .

What else is required of a well-ordered segment before we can be assured that it is partitionable ? Suppose it is not partitionable. Then there exists some $x, y \in S_{a,b}$ *such that* $y \notin S_{a,x} \cup S_{x,b}$. *Since line segments are WARNING NOT ALWAYS closed then we can find some* $\epsilon > 0$ *such that* $C_{x,\epsilon} \cap S_{a,x} == C_{x,\epsilon} \cap S_{x,b}$. *Without loss of generality take* $\rho(a,y) > \rho(a,x)$.

$$\forall k \in C_{x,\epsilon} \quad \rho(a,y) + \epsilon > \rho(a,k) > \rho(a,y) - \epsilon$$

If the segment is well-ordered, then $\nexists q \in S_{a,x} \mid \rho(a,q) = \rho(a,k)$. *(This relies on* $S_{a,x} \subset S_{a,b}$*, proven below.) This leaves a gap–it is continuous?*

We can use a weaker condition than partitionable . Call a line segment 'locally partitionable ' if, for all but a finite number of points, we have

$$x \in S_{a,b}, \exists \epsilon > 0 \mid (C_{x,\epsilon} \cap S_{a,x}) + (C_{x,\epsilon} \cap S_{b,x}) = C_{x,\epsilon} \cap S_{a,b}$$

A line segment can bifurcate, but so long as there are only a finite number of these bifurcation points it can be locally partitionable . That $S_{a,x} \subset S_{a,b}$ if $x \in S_{a,b}$ I prove in section 2.5. Along with this we can define the property of being 'locally well-ordered' in a natural way.

These various definitions are meant to help isolate what we mean when we talk about the 'width' of a line, and what makes a line thin.

2.5 Does $S_{a,b}$ Contain its Sub-segments?

Suppose that there is some point q in $S_{a,b}$. Is it true that $S_{a,q} \subset S_{a,b}$?

Suppose $\exists y \in S_{a,q}$. Then $\rho(a,y) + \rho(y,q) = \rho(a,q)$, and since $\rho(a,q) + \rho(q,b) = \rho(a,b)$, then $\rho(a,y) + \rho(y,q) + \rho(q,b) = \rho(a,b)$. Since we have, by definition of a metric space, $\rho(y,q) + \rho(q,b) \ge \rho(y,b)$, we get

$$\rho(a,b) = \rho(a,y) + \rho(y,q) + \rho(q,b) \ge \rho(a,y) + \rho(y,b) \ge \rho(a,b)$$

Since we are bounded above and below by $\rho(a,b)$ the $\ge$ must be $=$, and we see that $\rho(a,y) + \rho(y,b) = \rho(a,b)$, and hence $y \in S_{a,b}$.

Thus $S_{a,q} \subset S_{a,b}$ if $q \in S_{a,b}$.

In general, however, does a line segment contain its sub-segments? The answer is no, but it may be of interest to determine when it does and when it does not.

If for some x, y in $S_{a,b}$, $S_{x,y} \not\subset S_{a,b}$, then clearly x is not in $S_{a,y}$ or $S_{b,y}$, and likewise y is not in $S_{a,x}$ or $S_{b,x}$, or else by the result above $S_{x,y} \subset S_{a,y} \subset S_{a,b}$ (for example). Thus if $S_{x,y} \not\subset S_{a,b}$, then $S_{a,b} \neq S_{a,x} \cup S_{x,b}$, since we have a point y which is not in either of the two sub-segments. $S_{a,b}$ is not partitionable .

Notice that the converse is not true: $S_{a,b}$ not partitionable does NOT imply that there exists $\{x, y\} \in S_{a,b}$ with $S_{x,y} \not\subset S_{a,b}$. The 'city streets' metric space provides a simple counter-example: each line segment contains all segments createable from points within it, but it is not a simple sum of two sub-segments $S_{a,x}$ and $S_{x,b}$.

A line need not contain all line segments generated by the points within it. This is obvious from considering the 'great circle' metric, where a line is (in general) an arc extending more than half-way round the sphere. Thus there are two points $\{x, y\}$ in $L_{a,b}$ which are exactly opposite to each other, and $S_{x,y}$ is the entire surface of the sphere–which is NOT contained in $L_{a,b}$ in general.

The 'potential well' metric space offers an example of a space in which not all line segments are partitionable , and in fact in which there exist line segments which do not contain line segments generated from points within themselves. It is easy to see that most of the line segment arcs in this space are partitionable , but when the points are at different radii and are opposite each other in θ, the line segment consists of two non-circular arcs joining the points. Clearly the $S_{a,c}$ and $S_{c,b}$ formed from using a point in one of these arcs will not generate any points in the other arc, and this $S_{a,b}$ is not partitionable . In addition, if you take a point from one of the arcs and a point from the other, the line segment formed between them will, far from being a subset of $S_{a,b}$, only intersect $S_{a,b}$ in two points. This is clearly not a convex set, though most of the line segments in this metric space are convex.

2.6 Nearest Point in a Line Segment

Given a line segment $S_{a,b}$ (not equivalent to the entire space) and a point c not in it, then define the nearest points as

$$N(S_{a,b}, c) = \{x \in S_{a,b} \,|\, \rho(c, x) = d \,|\, d = \min(\rho(c, y)) \; y \in S_{a,b}\}$$

NOT ALWAYS TRUE Since $S_{a,b}$ is closed, there is at least one point in N.

There can be more than one point in N. For example take the 'great circle' metric space. Consider one $S_{a,b}$ which is an arc on the 'equator' and c on a 'pole'–In this case $N(S_{a,b}, c)$ is the entire $S_{a,b}$. The 'great circle' metric space was not devised to be pathological.

2.7 Intersection of Line Segments

Assume there are points a, b, c, and d such that $c, d \notin S_{a,b}$ and $a, b \notin S_{c,d}$. This will not always be possible, of course (as when a and b are on opposite poles of a sphere, with the usual metric on a sphere). Call their intersection

$$I_{a,b|c,d} \equiv S_{a,b} \cap S_{c,d}$$

Often I will be $\emptyset$, and there are metric spaces in which it is always $\emptyset$ (using the trivial metric, for example), but consider for now the instances when it is not empty, and also not the entire space (as can happen in 1-dimensional spaces).

WARNING Clearly I is closed. Given any $x, y \in I_{a,b|c,d}$, then $S_{x,y} \subset I_{a,b|c,d}$, since $S_{x,y}$ must be a subset of both of the original line segments.

Is $I_{a,b|c,d} = S_{r,s}$ for some r and s? Not always: there is a counterexample in the 'potential well metric.' Points exactly opposite each other in angle but at different radii have 2-branched line segments, which can have two intersection points with the segment joining a pair of points at the same radius. These don't form even a trivial line segment.

So instead of $I_{a,b|c,d}$, consider the connected subsets of it. Label these (if there are countably many) with i and call them $J^i_{a,b|c,d}$.

Define a 'diameter.' Let $D = \max(\rho(x, y))$ where $x, y \in J^i_{a,b|c,d}$. *WARNING about closure* Since J is closed, max is the same as sup, so $\exists m, n \in J^i_{a,b|c,d}$ which have $\rho(m, n) = D$. If $D = 0$, then $m = n$ and there is only one point in $J^i_{a,b|c,d}$, which one could then call $S_{m,m}$, and the conjecture is true trivially.

If $D > 0$, then there are one or more pairs of distinct points with $\rho(m, n) = D$. Let M be the set of all such pairs;

$$M \equiv \{(m, n) \mid m, n \in J^i_{a,b|c,d}\, ,\, \rho(m, n) = \max(\rho(x, y)) \mid x, y \in J^i_{a,b|c,d}$$

Order within a pair does not matter, and we require that each pair only appear once, to avoid double counting. Obviously our only candidates for the desired r and s are in the pairs in M.

As of this moment I have not determined the answer to the conjecture above. Perhaps one may have to classify metric spaces into those for which it is true and those for which it is not, and into those for which M may have more than one pair and those for which it never has more than one pair.

Assume for the moment that the union of the two lines is not the same as the entire space. Given a point i in the intersection, can we create a circle $C_{i:R} \equiv \{x \mid \rho(x, i) = R\}$ which contains points not in the intersection of the two lines? Assume we can, for R greater than some R_{min} (though the cases in which one cannot might have interesting pathologies). Now consider $U \equiv C_{i:R} \cap S_{a,b}$. Into how many continuous clumps is it divided?

If there are 0, 1 or more than 2 convex parts I'm not ready to deal with the situation right now. If there are 2, then let's proceed.

$U \equiv U_1 \cup U_2$ where U_1 and U_2 are the two convex parts. Let $p_1 \in U_1 \cap \overline{J^i_{a,b|c,d}}$ and $p_2 \in U_2 \cap \overline{J^i_{a,b|c,d}}$. If either of these two sets is empty, we again have a curious situation which I'll ignore for the moment. Now find a point Q (if it exists) in $S_{c,d}$ such that $Q \notin S_{a,b}$ and $\rho(Q, i) > R$. Let $W_1 = \min(\rho(p_1, Q)), p_1 \in U_1 \cap \overline{J^i_{a,b|c,d}}$, and W_2 be the corresponding minimum for p_2. Use these to define $f(Q, R) \equiv W_1^2/(W_1^2 + W_2^2)$. Now define $f_L(R) = \min(f(Q, R))$ and $f_H(R) = \max(f(Q, R))$. If these converge such that $\lim_{R\to R_{min}} f_H(R) - f_L(R) = 0$, then we can define a unique *angle* of intersection, which is given by $\sin(\theta/2) = \lim_{R\to R_{min}} f_H(R)$. It may be zero; perhaps in some metrics even always zero. For the standard Euclidean metric the definition returns the usual value for θ.

2.8 Dimensions

Dimensions aren't always easy to define, but I at least need to have something that allows me to exclude trivial cases. There are two obvious definitions of a 1-dimensional space: There exist two points a and b for which $L_{a,b}$ is the entire space; or alternatively, for all distinct points a and b, $L_{a,b}$ is the entire space. *It isn't clear which is most useful yet, and I have not taken up such fine points as "except for a finite number of points" or "except for a finite set of disconnected regions."*

2.9 Thick Line Segments

As noted before, under some metrics it is possible to find a point q in some line segments and a radius $r > 0$ such that the circle $C_{q:r}$ is a subset of the line segment. Let R be the maximum possible radius for some line segment, with $R > 0$. Since the distance between any two points in the line segment $S_{a,b}$ must be less than or equal to $2\rho(a, b)$, we have $R \leq 2\rho(a, b)$. However, with the 'city crow' metric, although you see that $S_{a,b}$ can have a kind of width, just like the 'taxi-cab' metric, it is never true that $C_{q:r}$ is a subset of the line segment if $r > 0$.

If $S_{a,b}$ contains all its $S_{x,y}$ where x and y are in $S_{a,b}$, then $\rho(x, y) \leq \rho(a, b)$. *At the moment I do not know if I can get tighter bounds in the general case or not.*

3 Planes

There are four obvious definitions of a 'plane' defined by 3 points. In the usual Euclidean space these are equivalent, but not in general.

First, one assumes that the three points a, b, and c are not in the same line. We can try to use a line defined by two of the points and a third point not on the line, as in

$$P^s_{L_{a,b},c} \equiv \{y \mid y \in L_{c,r};\ r \in L_{a,b}\} \tag{5}$$

Sometimes one will have $P^s_{L_{a,b},c} \equiv P^s_{L_{a,c},b} \equiv P^s_{L_{b,c},a}$, but this need not be true in general. A more symmetric definition is better:

$$P^2_{a,b,c} \equiv \{y \mid y \in L_{X,r};\ X \in \{a, b, c\},\ r \in L_{a,b} \cup L_{b,c} \cup L_{a,c}\} \tag{6}$$

In Euclidean geometry one can get away with an even smaller definition; though this might produce amusing unexpected gaps in the coverage:

$$P^1_{a,b,c} \equiv \{y \mid y \in L_{X,r};\ X \in \{a, b, c\},\ r \in S_{a,b} \cup S_{b,c} \cup S_{a,c}\} \tag{7}$$

Alternatively we can use a union of all lines generated from all points in the lines generated by the three points, as in

$$P^4_{a,b,c} \equiv \{y \mid y \in L_{s,r}; s, r \in L_{a,b} \cup L_{b,c} \cup L_{a,c}\} \tag{8}$$

Once again, in the Euclidean metric we can get away with a smaller definition involving lines between points on the line segments generated by the three points.

$$P^3_{a,b,c} \equiv \{y \mid y \in L_{s,r}; s, r \in S_{a,b} \cup S_{b,c} \cup S_{a,c}\} \tag{9}$$

We can also define a plane-like object by selecting one of the definitions of a 'plane' and generating all lines formed from points within that object; continuing the iteration until we get convergence (if that ever happens!).

An obvious first question is: 'Do these result in the same sets?' The answer is no. A simple counter-example is the 'great circle' metric. Given 3 points, the plane defined by P^1 consists, in general, of the area contained within 6 arcs defined by the points and lines between them–it is NOT the entire sphere, in general. However, P^4, consisting of all lines joining points in any of the lines, must consist of the entire sphere, since any line must include at least two points opposite each other on the sphere, and any line joining two points opposite each other comprises the entire sphere. Thus these are NOT equivalent definitions.

We have (by construction) that $P^s \subset P^2$ for any order of (a, b, c); and it is obvious that $P^1 \subset P^2$ and $P^3 \subset P^4$. It is not hard to see that $P^1 \subset P^3$ and $P^2 \subset P^4$, and if we iterate P^1 or P^3 as described above, that $P^2 \subset P^1_{iter}$ and $P^4 \subset P^3_{iter}$.

A second question is: 'Does a line partition a plane defined by that line and another point?' This depends on how one defines partition, apparently. In the case of the 'great circle' metric two points on 'opposite' sides of the line in a plane (P^1) defined by another point cannot be joined by a great circle arc which does not intersect the line, but can be joined by a series of arcs which don't intersect it. Of couse, P^1 is a deliberately minimal definition.

This needs more work.

4 Inside/Outside

Consider a 'triangle' defined by 3 points, none of which is in a line defined by the other two.

$$T_{a,b,c} \equiv \{\cup S_{i,j} \mid i, j \in \{S_{a,b} \cup S_{b,c} \cup S_{a,c}\}\} \tag{10}$$

Sometimes sweeping out the 'angles' from the vertices will be equivalent, but not always:

$$T^2{}_{a,b,c} \equiv \{\cup S_{i,j} \mid i \in \{a, b, c\}, j \in \{S_{a,b} \cup S_{b,c} \cup S_{a,c}\}\} \tag{11}$$

Clearly T^2 is a subset of T by construction. Cases in which it is a strict subset are rather curious.

If T^2 is a strict subset of T, without loss of generality we can say that a point $x \in T$ which is not in T^2 lies in $S_{q,r}$ where $q \in S_{a,b}$ and $r \in S_{a,c}$ where r and q are not amoung a, b, c.

Consider $L_{a,x}$. In a Euclidean triangle it would intersect $S_{b,c}$ somewhere. Suppose it does, and select $y \in L_{a,x} \cap S_{b,c}$. From the definition $\rho(a, x) + \rho(x, y) + \rho(a, y) = 2 \max(\rho(a, x), \rho(x, y), \rho(a, y))$.

If the maximum is $\rho(a,y)$, then $x \in S_{a,y}$ which contradicts the assumption that x is not in T^2. Therefore either the line though x and the 'vertex' a does not intersect the opposite line segment, or either the distance from the 'vertex' a to x or the distance from the 'intersection' y to x is *larger* than the distance from the 'vertex' to the 'intersection,' which is notably different from the Euclidean case, and makes understanding what is 'inside' and what is 'outside' a little complex.

What constitutes the inside and what the outside of the triangle? There may in fact be nothing 'inside' in any reasonable sense–for example consider the 'city streets' metric, in which the line segments between the three points completely fill the rectangle defined by the most extreme points. However, suppose we use the definition

$$In_{a,b,c} \equiv \{\cup S_{i,j} \mid i,j \in \{S_{a,b} \cup S_{b,c} \cup S_{a,c}\}\} - \{S_{a,b} \cup S_{b,c} \cup S_{a,c}\}$$

for the inside of this 'triangle', with the understanding that it may be empty.

I'd like to know if this definition of a triangle results in a triangle defined by three points which is a subset of the plane defined by those three points. It is consistent with P^3 and P^4, but it isn't clear yet if it works with the other two.

5 Miscellaneous Questions

Given a and b, under what conditions can one find x and y such that $S_{x,y}$ is $L_{a,b}$? This is certainly sometimes possible when the space is bounded and closed, as can be seen by considering the unit disk in R^2 with the standard 'crow flies' metric. When the space is unbounded, it is at least sometimes impossible for $L_{a,b}$ to be $S_{x,y}$.