Continuous Transformations Which Preserve the Structure of a Finite Group

James N. Bellinger

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Abstract

Some finite groups have a linear mapping of the group elements onto the algebra with the basis of the group elements such that the resulting linear combinations of group elements retain the group properties. In some cases these mappings are continuous Lie groups. So far I have found a 'non-compact' SO(3)xU(1), SO(4)xU(1), and Sl(2,c)xU(1).

UPDATE: March 2023 After some literature searches, I realized that the transformation in question is an isomorphism of a group algebra onto itself, and a fresh search found that this is a solved problem: "The group algebra of a finite group is symmetrically isomorphic to the direct sum of complete matrix algebras."

This means, among other things, that the initial question of 40 years ago, namely "If a finite group can model particle interactions, will symmetries over its algebra produce $SU(3)\times SU(2)\times U(1)$?" Answer: no–at least not for anything reasonably sized. Not that non-abelian groups were likely to be a useful model for particle interactions anyway.

1 Introduction and Physics Motivation

Particles such as the K_L we describe as a combination of K^0 and anti-K^0. Particles such as the π^0 are regarded as linear combinations of quarks and anti-quarks. It has been proposed that the particles we know or infer (leptons and quarks) are similarly composed of combinations of "preons."

I examine in this paper some consequences of a simple preon model, and show that certain continuous symmetries result.

The model I use has seven basic assumptions:

• First suppose that all preon interactions are 3-body interactions. I know of no 4-body interaction which has been observed, now that beta decay is known to progress through a series of 3-body interactions. A 4-gluon vertex is hypothesised, but not directly observed.

• Suppose secondly that all particles are created from combinations of these preons. This condition is not required for the mathematics of the model, but for any possible interpretation of it.

• Suppose thirdly that we are not for the moment concerned with the particle positions or momenta, but only with particle type. We now are treating all particle interactions as being of the form $A \bigodot B \rightarrow C$.

• Fourthly suppose that every particle may interact with every other to make a new particle. This requires that the particles in the model not have any net charge (as in $(e^+ + e^-)/\sqrt{2}$ vs $(e^+ - e^-)/\sqrt{2}$). I include no mechanism for giving the particles mass: all are assumed massless.

• Suppose fifthly that the interaction we have defined so far is associative: $A \bigodot (B \bigodot C) = (A \bigodot B) \bigodot C$. We can argue that associativity is a consequence of some of the conservation laws, though tighter conditions could be applied. This condition is deliberately loose.

• Suppose sixthly that we have some particle Q which has the property that $Q \bigodot A \rightarrow A$. We do in fact observe interactions with this property, such as an electron absorbing a photon. Remember that by assumption 3 we are dealing only with particle character or identity, not momentum.

• Suppose seventhly that for each particle A there is an 'anti-particle' A^{-1} for which $A \bigodot A^{-1} \rightarrow Q$. We have defined a group structure, which we may take to be finite.

Now I ask: What continuous symmetries exist in this model? This is something of a reversal of the standard approach which examines representations of a group to find particles. I am looking at a finite group representing preon interactions and searching for the observed continuous symmetry. Eigenvectors of such a transformation could be candidates for identification with observed particles.

The specific question I wish to address is this: Can the observed symmetries (SU(3), SU(2), etc.) be generated in a natural way from linear transformations over finite groups, where the structure of the finite group is maintained?

The answer, as I will show, is: Yes, some of them can be generated–perhaps all, I don't know. The price is high, however–the finite groups which represent particle interactions must be non-abelian. This is contrary to observation and intuition, but in one case explored in detail the eigenvectors of the transformation which have constant eigenvalues do in fact commute, and only those eigenvectors whose eigenvalues are functions of the parameters fail to commute. This suggests that only those particles whose preon contents are subject to change are those which fail to commute. This remains to be proven or disproven in general, however.

A second, as yet unanswered, question is: Can one predict which symmetries can be generated? There is a hint that by examining the group characters one can determine the order of continuous symmetries, but this hasn't been fully explored yet.

2 Foundations

The method is to consider the elements of a finite group as the basis of a vector space, and then study those transformations of the basis elements for which the new bases, considered as elements of a new finite group themselves, form a group isomorphic to the original. Each of the new bases is a linear combination (complex coefficients) of the original basis elements (elements of the group).

Consider a finite group G with elements c_i. Number them in some arbitrary order, from 0 to $N-1$, where 0 is the label for the identity element and N is the order of the group. Display the group operation by $c_i \bigodot c_j \to c_k$.

I work with a space in which the elements P are defined as $\sum_i p_i\, c_i$, where p_i is a complex number (it could be some other field, but I haven't addressed that question), and c_i are the elements of the original finite group. Multiplication by a complex number is naturally defined by $b\,P$ as $\sum_i (b\,p_i)\,c_i$, and the sum of two elements P and Q is $\sum_i (p_i + q_i)\,c_i$. There is also a natural operation between the elements of this space given in terms of the group operation by $P \bigotimes Q = \sum_i \sum_j p_i q_j (c_i \bigodot c_j)$.

In this space consider a linear transformation of the original group elements, so that we have a new group G' with $c'_i \bigotimes c'_j \to c'_k$. This is an automorphism over the new space. Let the array V be defined by $c'_i = V_{i,r} c_r$ so we can write

$$V_{i,r} c_r \bigotimes V_{j,s} c_s \to \phi_k V_{k,t} c_t \tag{1}$$

$$V_{i,ts^{-1}} c_{ts^{-1}} \bigotimes V_{j,s} c_s \to V_{ij,t} c_t \tag{2}$$

$$V_{i,ts^{-1}} c_{ts^{-1}} \bigotimes V_{j,s} c_s \to \phi_{ij} V_{ij,t} c_t \tag{3}$$

$$V_{i,ts^{-1}}\, V_{j,s} = \phi_{ij} V_{ij,t} = V_{i,r}\, V_{j,r^{-1}t} \tag{4}$$

Note that I assume that V is complex. One could restrict it to be real, or use some other field entirely, but I'll take the complex case. It turns out that the ϕ_{ij} are all identical, and can be divided out as a completely independent phase (U(1), in other words). In what follows I will denote the identity by 0, for simplicity and clarity. The top element of a column vector will correspond to this 0-element. In what follows, if an object has more than one subscript, these will be separated by commas. If several terms are concatenated in a subscript without commas, group addition of the specified elements is assumed. First let me demonstrate that the ϕ_x are identical. Clearly V must not be singular, or the resulting transformed group would be smaller than the original.

Define $f_i = \sum_x V_{i,x}$. Then since

$$\sum_r V_{i,r} V_{j,r^{-1}t} = V_{ij,t} \phi_{ij}$$

$$\sum_{r,t} V_{i,r} V_{j,r^{-1}t} = f_i f_j = \sum_t V_{ij,t} \phi_{ij} = \phi_{ij} f_{ij}$$

When $i = 0$, we get $f_0\, f_j = \phi_j\, f_j$. If any f_k were 0, all would be, and thus V would be singular. Since V is non-singular, $f_j \neq 0$, so $f_0 = \phi_j \forall j$, thus all the ϕ_j are the same, and will be called simply ϕ, which is also clearly non-zero.

To continue, restate the equation of f's as $f_i\, f_j = f_{ij}\, \phi$. Then $f_{i^2} = f_i^2/\phi$. This extends to $f_{i^n} = f_i^n (1/\phi)^{n-1}$. Now we know that for each i an element in the group there exists some N such that $i^N = 0$ (the identity), since this is a finite group. Then $f_{i^N} = f_0 = \phi = f_i^N (1/\phi)^{N-1}$, or, rearranging, $f_i^N = \phi^N$. Thus

$$f_i = \phi e^{2i\pi \frac{n_i}{N}} \qquad 0 \leq n_i \leq N$$

But notice that the n_i are discrete, and this is a continuous transformation. The identity is obviously a member of the set of transformations starting from the identity, and for the identity each f_i is 1. If we parameterize starting from the identity transform, then each n_i must be 0. Therefore,

$$\sum_x V_{i,x} = f_i = \phi \quad \forall i$$

Let $g_i = \sum_x V_{x,i}$. Then

$$\sum_{i,r} V_{i,r}\, V_{j,r^{-1}t} = \sum_t V_{ij,t} \phi = \sum_r g_r\, V_{j,r^{-1}t} = g_t \phi$$

$$\sum_s V_{j,s} g_{s^{-1}} = g_0 \phi \quad \forall j$$

Let $\mathbf{G}_s = \{g_{s^{-1}}\}$, a column vector. If $\mathbf{1}$ is defined as a column vector of ones, then we may write $V\mathbf{G} = g_0 \phi \mathbf{1}$, which may be solved with $\mathbf{G} = g_0 \phi V^{-1} \mathbf{1}$. Note that V^{-1} is also a transformation, and the sum of elements in a row in it is equal to some other ϕ', which may be readily seen to be ϕ^*. Thus, $\mathbf{G} = g_0 \phi \phi^* \mathbf{1}$, or $\mathbf{G} = g_0 \mathbf{1}$. Thus $g_i = g_0 \forall i$.

$$N g_0 = \sum_i g_i = \sum_{i,x} V_{x,i} = \sum_{x,i} V_{x,i} = \sum_x f_x = N\phi \Rightarrow g_0 = \phi$$

For convenience, let us divide V by ϕ, so that we can quit writing it out each time. Let $V = V'\, \phi$. Then

$$V_{i,r}\, V_{j,r^{-1}t} = V'_{i,r}\, V'_{j,r^{-1}t}\, \phi^2 = V_{ij,t} \phi = V'_{ij,t} \phi^2$$

$$V'_{i,r}\, V'_{j,r^{-1}t} = V'_{ij,t}$$

In what follows I will assume that the ϕ has been divided out, and that the **new** equations (dropping the primes) governing the transformation are

$$V_{i,r}\, V_{j,r^{-1}t} = V_{ij,t} \qquad \sum_i V_{i,j} = \sum_j V_{i,j} = 1 \tag{5}$$

2.1 Infinitesimal Transformations

Now consider infinitesimal transformations away from the identity. Here $s = r^{-1}t$.

$$V_{i,ts^{-1}}V_{j,s} = V_{ij,t}$$

$$V_{a,b} \Rightarrow \delta_{a,b} + \delta V_{a,b} \tag{6}$$

$$(\delta_{i,ts^{-1}} + \delta V_{i,ts^{-1}})(\delta_{j,s} + \delta V_{j,s}) = \delta_{ij,t} + \delta V_{ij,t}$$

$$\delta_{i,ts^{-1}}\delta_{j,s} + \delta_{j,s}\delta V_{i,ts^{-1}} + \delta_{i,ts^{-1}}\delta V_{j,s} + O(\delta^2) = \delta_{ij,t} + \delta V_{i,tj^{-1}} + \delta V_{j,i^{-1}t} = \delta_{ij,t} + \delta V_{ij,t}$$

resulting in

$$\delta V_{i,tj^{-1}} + \delta V_{j,i^{-1}t} = \delta V_{ij,t} \quad , \quad \sum_i \delta V_{i,j} = \sum_j \delta V_{i,j} = 0 \tag{7}$$

The above are the fundamental equations governing the transformations of the array. Let $i = j = 0$. Then the fundamental equation reduces to $2\delta V_{0,t} = \delta V_{0,t}$, so

$$\delta V_{0,t} = 0 \tag{8}$$

Let $t = j$. Then the fundamental equation becomes

$$\delta V_{i,0} + \delta V_{j,i^{-1}j} = \delta V_{ij,j} \tag{9}$$

Now for any $i \neq 0$ we know that there exists some $N \geq 2$ such that $i^N = 0$, where 0 is the identity. Thus we make the following substitutions

$$j = i \quad \Rightarrow \quad 2\delta V_{i,0} = \delta V_{i^2,i} \tag{10}$$

$$j = i^2 \quad \Rightarrow \quad \delta V_{i,0} + \delta V_{i^2,i} = \delta V_{i^3,i^2} = 3\delta V_{i,0} \tag{11}$$

$$j = i^N \quad \Rightarrow \quad N\delta V_{i,0} = \delta V_{i^N,i^{N-1}} = \delta V_{0,i^{N-1}} \tag{12}$$

$$\tag{13}$$

But since $\delta V_{0,x} = 0$, we must have

$$\delta V_{i,0} = 0 \;\; \forall i \tag{14}$$

The column vector in δV corresponding to this is all zero's ($\delta 1 = 0$ for the (0,0) position), and the top row is also all zero's. Remember that I am placing the identity element in the first position. We formally express V as $exp(\sum c_x \delta V_x)$ where the δV_x are independent infinitesimal transformations away from the initial value. Therefore, since the initial value of V is the identity, which has zeros off the diagonal element in this row and column, any product of any δV's with the initial value of V must continue to have zero's in these positions. Thus all V's have $V_{0,0} = 1$, and $V_{x,0} = V_{0,x} = 0$.

$$V = \begin{pmatrix} 1 & 0 \ldots 0 \\ 0 & \\ \vdots & V' \\ 0 & \end{pmatrix} \tag{15}$$

Therefore, the identity element does not transform.

From the fundamental rule governing the continuous transformations,

$$\delta V_{i,tj^{-1}} + \delta V_{j,i^{-1}t} = \delta V_{ij,t}$$

if we let $t = 0$ we have

$$\delta V_{i,j^{-1}} + \delta V_{j,i^{-1}} = 0$$

or

$$\delta V_{i,q} = -\delta V_{q^{-1},i^{-1}} \quad \forall i, q \tag{16}$$

Now restate the first fundamental equation, substituting $t = jq$.

$$\delta V_{i,jqj^{-1}} + \delta V_{j,i^{-1}jq} = \delta V_{ij,jq}$$

If j commutes with all other elements of the group, then $jqj^{-1} = q$, and $ij = ji$, so

$$\delta V_{i,q} + \delta V_{j,ji^{-1}q} = \delta V_{ji,jq}$$

Now consider the sequence of equations generated by substituting for $i = jI$ and $q = jQ$.

$$\delta V_{j^2I,j^2Q} = \delta V_{j,ji^{-1}q} + \delta V_{jI,jQ} = \delta V_{j,ji^{-1}q} + \delta V_{I,Q} + \delta V_{j,ji^{-1}q} = 2\delta V_{j,ji^{-1}q} + \delta V_{I,Q}$$

$$\delta V_{j^3I,j^3Q} = 3\delta V_{j,ji^{-1}q} + \delta V_{I,Q}$$

and so on. For some N, $j^N = 0$, and we get

$$\delta V_{j^NI,j^NQ} = N\delta V_{j,ji^{-1}q} + \delta V_{I,Q} = \delta V_{I,Q}$$

from which we get $\delta V_{j,ji^{-1}q} = 0$. Since this is true for arbitrary i and q, we see that $\delta V_{j,x} = 0\,\forall x$. Now since j^{-1} will also commute, the same reasoning applied to it, and we have $\delta V_{j^{-1},x} = 0\,\forall x$, which implies $\delta V_{x^{-1},j} = 0\,\forall x$. Thus **if j commutes with all other elements of the group, j is not transformed.**

In an abelian group, any j commutes with all other elements, so the above is true for all j in the group. Thus **abelian groups are not continuously transformable**. One may further state that Abelian sub-groups do not "internally" transform: the matrix elements of the δV array which connect one element in an abelian sub-group with another element in the subgroup are 0. This is trivial for a subgroup of order 2.

Clearly if this model does correspond to a physical system, it is not an immediately intuitive one.

2.2 Further Simplifications

Now let us look at diagonal elements. In the fundamental equation of transformation, above, substitute $j = i$ and $t = i^2$. We then get $2\delta V_{i,i} = \delta V_{i^2,i^2}$. If $j = i^2$, then we find $\delta V_{i,i} + 2\delta V_{i^2,i^2} = \delta V_{i^3,i^3}$, and so on. This gives $N\delta V_{i,i} = \delta V_{i^N,i^N}$, but since for some N, $i^N = 0$, we must have

$$\delta V_{i,i} = 0 \quad \forall i \tag{17}$$

In another look at the fundamental equation, set $t = j$. This results in

$$\delta V_{ij,j} = \delta V_{j,i^{-1}j} \tag{18}$$

Alternatively, if we set $ij \equiv q$, then we have

$$\delta V_{i,tq^{-1}i} + \delta V_{i^{-1}q,i^{-1}t} = \delta V_{q,t} \tag{19}$$

If $t = i$, then

$$\delta V_{i,iq^{-1}i} = \delta V_{q,i} \tag{20}$$

Instead of reducing, we can use the inversion derived earlier ($\delta V_{j,r} = -\delta V_{j^{-1},r^{-1}}$), and find that $-\delta V_{i^{-1}qt^{-1},i^{-1}} + \delta V_{i^{-1}q,i^{-1}t} = \delta V_{q,t}$. Since i is arbitrary, substitute j for i^{-1} to find

$$\delta V_{jq,jt} - \delta V_{jqt^{-1},j} = \delta V_{q,t} \quad \forall j \tag{21}$$

These equations are powerful tools, since they say that for any q and t one has N-1 combinations of matrix elements all equal to the single matrix element $\delta V_{q,t}$.

2.3 Conjugacy Classes and Eigenvectors

Each element f in the group is a member of some conjugacy class, which is the set F of all elements in the group such that if r is in F, then there exists some g in the group for which $g\, r\, g^{-1} = f$. One interesting question (motivated by looking at a few examples) is 'How does the sum of members of a conjugacy class transform?' The answer, as is shown below, is that such a sum does remains the same under the differential transformation used so far.

The fundamental equation is

$$\delta V_{i,tj^{-1}} + \delta V_{j,i^{-1}t} = \delta V_{ij,t}$$

Substitute for t the quantity $j^2\, s\, j^{-1}$, and sum the s over all members of its particular conjugacy class, which I'll call S. The result is:

$$\sum_{s\in S} \delta V_{i,j^2sj^{-2}} + \sum_{s\in S} \delta V_{j,i^{-1}j^2sj^{-1}} = \sum_{s\in S} \delta V_{ij,j^2sj^{-1}}$$

Since we are summing over all members of a conjugacy class, we can replace j^2sj^{-2} in the first term with s, $i^{-1}j^2sj^{-1}$ in the second term with $i^{-1}js$, and j^2sj^{-1} on the right side with js. This results in the much simpler

$$\sum_{s\in S} \delta V_{i,s} + \sum_{s\in S} \delta V_{j,i^{-1}js} = \sum_{s\in S} \delta V_{ij,js}$$

Now if we set $j = i$, we find

$$2\sum_{s\in S} \delta V_{i,s} = \sum_{s\in S} \delta V_{i^2,is}$$

For $j = i^2$, we get

$$\sum_{s\in S} \delta V_{i,s} + \sum_{s\in S} \delta V_{i^2,is} = \sum_{s\in S} \delta V_{i^3,i^2s} = 3\sum_{s\in S} \delta V_{i,s}$$

If $n > 0$, we have by induction for $j = i^n$

$$\sum_{s \in S} \delta V_{i^{n+1}, i^n s} = (n+1) \sum_{s \in S} \delta V_{i,s}$$

For some m, $i^m = 0$, so for $n = m - 1$

$$\sum_{s \in S} \delta V_{i^m, i^{m-1} s} = m \sum_{s \in S} \delta V_{i,s}$$

But since $\delta V_{0,x} = 0$, the left side is 0, and thus

$$\sum_{s \in S} \delta V_{i,s} = 0$$

These sums of elements in a conjugacy class are thus eigenvectors of the transformation V, with constant eigenvalue 1. Such sums of elements not only do not transform under V, but they commute with any other linear combination of elements of the group.

2.4 Discrete Transforms: Permutations

We can have transformations which are simply permutations of the group elements, as well as permutations with a sign. So long as these preserve the group structure they are legitimate objects of study here. Some of these permutations may arise naturally from continuous transformations from the identity, but some do not. Those which do not arise from transformations from the form a finite group themselves, and each member of this set of group-preserving permutations may serve as the basis for a family of continuous transformations over the original group.

Clearly elements of a conjugacy class must either map into each other or into elements of a conjugacy class of the same size. This helps restrict the number of cases to examine.

Let's returning to our physical model–preons with no well-defined quantum numbers which are combined to form objects which DO have well-defined quantum numbers. Presumably the constant eigenvectors of the transformation correspond to real particles, or things from which real particles could be generated. Since most of the constant eigenvectors are sums of elements ('preons') from the same 'conjugacy class', it isn't clear that we have a natural way to find anti-particles from preons without the use of 'signed permutation transforms'.

In any case the model is becoming somewhat unwieldy, with groups up to order 16 studied without finding the SU(3) color group. A preon model with more preons than particles is unaesthetic, not to mention dubious.

3 Simple Examples

I am a firm believer in the power of examples, and will generate a number of them for use in checking hypotheses.

Consider the group of the symmetries of an equilateral triangle. It has 6 elements. O_3

We may designate the elements of the group by *0* , *1* , *2* , *3* , *4* , and *5* , where *0* is the identity, $a \equiv 4$, $b \equiv 1$, $b^2 \equiv 2$, and so on. We may define a product (Cayley) table for this group in the following way:

$$\begin{array}{c|cccccc} \odot & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 0 & 4 & 5 & 3 \\ 2 & 2 & 0 & 1 & 5 & 3 & 4 \\ 3 & 3 & 5 & 4 & 0 & 2 & 1 \\ 4 & 4 & 3 & 5 & 1 & 0 & 2 \\ 5 & 5 & 4 & 3 & 2 & 1 & 0 \end{array} \tag{22}$$

I will dispense with the labels above and to the left, since the group operation with *0* easily identifies which element is which.

It may be shown, via fairly tedious and trivial algebra, that all elements of δV are zero except a few which are all simply related to each other. If there are some tiny changes α, β, and γ; then

$$\delta V = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & -\beta & -\alpha+\beta \\ 0 & 0 & 0 & -\alpha & \beta & \alpha-\beta \\ 0 & \alpha & -\alpha & 0 & \gamma & -\gamma \\ 0 & -\beta & \beta & -\gamma & 0 & \gamma \\ 0 & -\alpha+\beta & \alpha-\beta & \gamma & -\gamma & 0 \end{pmatrix} \tag{23}$$

$$\delta V \equiv \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} \alpha + \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \beta + \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \gamma \tag{24}$$

where

$$a \equiv \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix} \quad , \quad b \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \quad b^T = b$$

$$c \equiv \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \quad c^T = -c$$

$$A = \begin{pmatrix} 0 & a \\ a^T & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & b \\ b^T & 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \tag{25}$$

The eigenvectors corresponding to eigenvalue 0 of δV are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ \gamma \\ -\gamma \\ \alpha-\beta \\ \beta-\alpha \\ \alpha+\beta \end{pmatrix}$$

Notice that these eigenvectors commute under the defined field, as expected, as they are sums of the elements in the conjugacy classes.

$$[A, B] = -2C \qquad [B, C] = 2A + B \qquad [A, C] = -A - 2B$$

If we set

$$X \equiv \frac{1}{2}(A + B) \qquad Y \equiv \frac{1}{\sqrt{3}}C \qquad Z \equiv \frac{1}{2\sqrt{3}}(A - B)$$

Then we find

$$[X, Y] = Z \qquad [Y, Z] = X \qquad [X, Z] = Y$$

Let us now define

$$e \equiv A\cos\theta + B + C\sin\theta \qquad f \equiv A\cos\theta - B + C\sin\theta \qquad h \equiv A2\sin\theta - B2\cos\theta$$

The resulting commutation relations are

$$[e, f] = h \qquad [h, e] = 2e \qquad [h, f] = -2f$$

which are the conditions for generating the simplest Kac-Moody algebra [1]. If we set

$$T_1 \equiv \frac{1}{2}(A + B) \qquad T_2 \equiv \frac{1}{2\sqrt{3}}(A - B) \qquad T_3 \equiv \frac{-i}{\sqrt{3}}C$$

then we have $[T_i, T_j] = i\epsilon_{ijk}T_k$ this is related to the generators for SU(2) or SO(3).

Look at powers of δV. The second power is

$$\delta V^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2(\alpha^2+\beta^2-\alpha\beta) & -2(\alpha^2+\beta^2-\alpha\beta) & \gamma(2\beta-\alpha) & \gamma(2\alpha-\beta) & -\gamma(\alpha+\beta) \\ 0 & -2(\alpha^2+\beta^2-\alpha\beta) & 2(\alpha^2+\beta^2-\alpha\beta) & \gamma(\alpha-2\beta) & -\gamma(2\alpha-\beta) & \gamma(\alpha+\beta) \\ 0 & \gamma(\alpha-2\beta) & \gamma(2\beta-\alpha) & 2(\alpha^2-\gamma^2) & -2\alpha\beta+\gamma^2 & -2(\alpha^2-\alpha\beta)+\gamma^2 \\ 0 & -\gamma(2\alpha-\beta) & \gamma(2\alpha-\beta) & -2\alpha\beta+\gamma^2 & 2(\beta^2-\gamma^2) & -2(\beta^2-\alpha\beta)+\gamma^2 \\ 0 & \gamma(\alpha+\beta) & -\gamma(\alpha+\beta) & 2(-\alpha^2+\alpha\beta)+\gamma^2 & -2(\beta^2-\alpha\beta)+\gamma^2 & 2(\alpha^2-2\alpha\beta+\beta^2)-2\gamma^2 \end{pmatrix} \tag{26}$$

To find V, we use $V = e^{\delta V}$.

The eigenvalues of the array δV are given by

$$0 = X^4\left(4\alpha^2 - 4\alpha\beta + 4\beta^2 - 3\gamma^2 - X^2\right) \tag{27}$$

[1] Infinite Dimensional Lie Algebras, Victor G. Kac, page x; suggested by Georgia Benkart

The eigenvalues of $\exp(\delta V)$ thus have 4 $1'$s, and 2 others which may be real or imaginary depending on α, β, and γ.

There can be transformations which consist of simple permutations of the elements, or of permutations with a sign. For this group there are six unsigned permutations, which may or may not be special cases of the above continuous transform. For those which are **not** special cases, the continuous transform may be simply applied to these permutations to get new families of transformations.

The first set of permutations is given by the mapping $0 \to 0$, $1 \to 1$, $2 \to 2$, and the following 3 cases: The identity ($3 \to 3$, $4 \to 4$, and $5 \to 5$), a left rotation ($3 \to 4$, $4 \to 5$, and $5 \to 3$), and a right rotation ($3 \to 5$, $4 \to 3$, and $5 \to 4$). These can be derived from transformations from the identity.

Another set of permutations is also reachable from transformations from the identity: $0 \to 0$, $1 \to 2$, and $2 \to 1$, together with the following 3 cases: ($3 \to 3$, $4 \to 5$, and $5 \to 4$), ($3 \to 4$, $4 \to 3$, and $5 \to 5$), and ($3 \to 5$, $4 \to 4$, and $5 \to 3$).

Consider the group of the symmetries of a square. It has 8 elements, and may be defined by the following equations:

$$a^2 \to 0 \quad , \quad b^4 \to 0 \quad , \quad ba \to ab^3 \tag{28}$$

We may designate the elements of the group by *0* , *1* , *2* , *3* , *4* , *5* , *6* , and *7* , where *0* is the identity, $a \equiv 4$, $b \equiv 1$, $b^2 \equiv 2$, and so on. We may define a product (Cayley) table for this group in the following way:

⊙	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
0	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1	*1*	*2*	*3*	*0*	*5*	*6*	*7*	*4*
2	*2*	*3*	*0*	*1*	*6*	*7*	*4*	*5*
3	*3*	*0*	*1*	*2*	*7*	*4*	*5*	*6*
4	*4*	*7*	*6*	*5*	*0*	*3*	*2*	*1*
5	*5*	*4*	*7*	*6*	*1*	*0*	*3*	*2*
6	*6*	*5*	*4*	*7*	*2*	*1*	*0*	*3*
7	*7*	*6*	*5*	*4*	*3*	*2*	*1*	*0*

(29)

In the future I will dispense with the labels above and to the left, since the group operation with *0* easily identifies which element is which.

It may be shown, via fairly tedious and trivial algebra, that all elements of δV are zero except a few which are all simply related to each other. Let there be some tiny changes α, β, and γ; then

$$\delta V = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & \beta & -\alpha & -\beta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha & -\beta & \alpha & \beta \\ 0 & \alpha & 0 & -\alpha & 0 & \gamma & 0 & -\gamma \\ 0 & \beta & 0 & -\beta & -\gamma & 0 & \gamma & 0 \\ 0 & -\alpha & 0 & \alpha & 0 & -\gamma & 0 & \gamma \\ 0 & -\beta & 0 & \beta & \gamma & 0 & -\gamma & 0 \end{pmatrix} \tag{30}$$

$$\delta V \equiv \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} 2\alpha + \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} 2\beta + \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} 2\gamma \tag{31}$$

where

$$A \equiv \frac{1}{2}\begin{pmatrix} a & -a \\ -a & a \end{pmatrix} \qquad a \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$B \equiv \frac{1}{2}\begin{pmatrix} b & -b \\ -b & b \end{pmatrix} \qquad b \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad B^T = B$$

$$C \equiv \frac{1}{2}\begin{pmatrix} c & -c \\ -c & c \end{pmatrix} \qquad c \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad C^T = -C$$

$$\delta V^3 = 4\left(\alpha^2 + \beta^2 - \gamma^2\right)\delta V \tag{32}$$

Notice that

$$[A, B] = C \qquad [C, B] = A \qquad [A, C] = B$$

Let $X = A$, $Y = C$, and $Z = B$. Then

$$[X, Y] = Z \qquad [Y, Z] = X \qquad [X, Z] = Y$$

and this has the same structure as the 'triangle group'.

To find V, we use $V = e^{\delta V}$.

The eigenvalues of the δV array are given by

$$0 = X^6\left(X^2 - 4\left(\alpha^2 + \beta^2 - \gamma^2\right)\right) \tag{33}$$

The eigenvalues of $\exp(\delta V)$ have then 6 1's and two others which are either real or imaginary, depending on the values of α, β, and γ.

4 An Example with the Quaternion Group

Consider the quaternion group. It has 8 elements, and may be defined by the following equations:

$$a^4 \to 0 \quad , \quad b^2 \to a^2 \quad , \quad ba \to a^3 b \tag{34}$$

We may designate the elements of the group by *0* , *1* , *2* , *3* , *4* , *5* , *6* , and *7* , where *0* is the identity, $a^2 \equiv 1$, $a \equiv 2$, $a^3 \equiv 3$, $b \equiv 4$, $a^2 b \equiv 5$, $ab \equiv 6$, and $a^3 b \equiv 7$. We may define a product (Cayley) table for this group in the following way:

$$\begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\ 2 & 3 & 1 & 0 & 6 & 7 & 5 & 4 \\ 3 & 2 & 0 & 1 & 7 & 6 & 4 & 5 \\ 4 & 5 & 7 & 6 & 1 & 0 & 2 & 3 \\ 5 & 4 & 6 & 7 & 0 & 1 & 3 & 2 \\ 6 & 7 & 4 & 5 & 3 & 2 & 1 & 0 \\ 7 & 6 & 5 & 4 & 2 & 3 & 0 & 1 \end{matrix} \tag{35}$$

Grinding through the algebra will show that all elements of δV are zero except a few which are described by three independent tiny changes α, β, and γ.

$$\delta V = \alpha \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & -\alpha & \beta & -\beta \\ 0 & 0 & 0 & 0 & -\alpha & \alpha & -\beta & \beta \\ 0 & 0 & -\alpha & \alpha & 0 & 0 & \gamma & -\gamma \\ 0 & 0 & \alpha & -\alpha & 0 & 0 & -\gamma & \gamma \\ 0 & 0 & -\beta & \beta & -\gamma & \gamma & 0 & 0 \\ 0 & 0 & \beta & -\beta & \gamma & -\gamma & 0 & 0 \end{pmatrix} \tag{36}$$

$$\text{Let} \quad a \equiv \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad a^2 = a$$

$$A \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad B \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \end{pmatrix} \qquad C \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a^2 & 0 \end{pmatrix} \qquad AB = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$AB - BA = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a^2 \\ 0 & 0 & a^2 & 0 \end{pmatrix} = -C$$

$$AC - CA = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a^2 \\ 0 & 0 & 0 & 0 \\ 0 & -a^2 & 0 & 0 \end{pmatrix} = B$$

$$BC - CB = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -a^2 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -A$$

From the above, if we let

$$T_1 \equiv B \quad T_2 \equiv A \quad T_3 \equiv C$$

we see that

$$[T_i, T_j] = \epsilon_{ijk} T_k$$

which describes the generators for the groups O(3) and SU(2). At this point it is not obvious which is generated here. I cannot use real coefficients to get this into Kac-Moody form, though I can do it with complex ones ($h = 2iB$, $e = iA + C$, and $f = iA - C$).

To generate a transformation one may use the following method:

$$V = e^{\alpha A + \beta B + \gamma C}$$

$$\alpha A + \beta B + \gamma C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & -\alpha & 0 & \gamma \\ 0 & -\beta & -\gamma & 0 \end{pmatrix} \equiv Q$$

$$Q^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -(\alpha^2+\beta^2) & -\beta\gamma & \alpha\gamma \\ 0 & -\beta\gamma & -(\alpha^2+\gamma^2) & -\alpha\beta \\ 0 & \alpha\gamma & -\alpha\beta & -(\beta^2+\gamma^2) \end{pmatrix} \qquad Q^3 = -Q\left(\alpha^2+\beta^2+\gamma^2\right)$$

In the above, every non-zero matrix element is understood to be multiplied by a, where a is the 2×2 array defined above.

$$V(\alpha,\beta,\gamma) = I + \frac{Q}{r}\sin r + \frac{Q^2}{r^2}(1-\cos r) \quad \text{where} \quad r \equiv \sqrt{\alpha^2+\beta^2+\gamma^2} \tag{37}$$

Define $\epsilon \equiv (1-\cos r)$, and $\delta \equiv \sin r$. In its full glory, the array expands to the following:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1-\frac{\alpha^2+\beta^2}{2r^2}\epsilon & \frac{\alpha^2+\beta^2}{2r^2}\epsilon & \frac{\alpha}{2r}\delta-\frac{\beta\gamma}{2r^2}\epsilon & -\frac{\alpha}{2r}\delta+\frac{\beta\gamma}{2r^2}\epsilon & \frac{\beta}{2r}\delta+\frac{\alpha\gamma}{2r^2}\epsilon & -\frac{\beta}{2r}\delta-\frac{\alpha\gamma}{2r^2}\epsilon \\
0 & 0 & \frac{\alpha^2+\beta^2}{2r^2}\epsilon & 1-\frac{\alpha^2+\beta^2}{2r^2}\epsilon & -\frac{\alpha}{2r}\delta+\frac{\beta\gamma}{2r^2}\epsilon & \frac{\alpha}{2r}\delta-\frac{\beta\gamma}{2r^2}\epsilon & -\frac{\beta}{2r}\delta-\frac{\alpha\gamma}{2r^2}\epsilon & \frac{\beta}{2r}\delta+\frac{\alpha\gamma}{2r^2}\epsilon \\
0 & 0 & -\frac{\alpha}{2r}\delta-\frac{\beta\gamma}{2r^2}\epsilon & \frac{\alpha}{2r}\delta+\frac{\beta\gamma}{2r^2}\epsilon & 1-\frac{\alpha^2+\gamma^2}{2r^2}\epsilon & \frac{\alpha^2+\gamma^2}{2r^2}\epsilon & \frac{\gamma}{2r}\delta-\frac{\alpha\beta}{2r^2}\epsilon & -\frac{\gamma}{2r}\delta+\frac{\alpha\beta}{2r^2}\epsilon \\
0 & 0 & \frac{\alpha}{2r}\delta+\frac{\beta\gamma}{2r^2}\epsilon & -\frac{\alpha}{2r}\delta-\frac{\beta\gamma}{2r^2}\epsilon & \frac{\alpha^2+\gamma^2}{2r^2}\epsilon & 1-\frac{\alpha^2+\gamma^2}{2r^2}\epsilon & -\frac{\gamma}{2r}\delta+\frac{\alpha\beta}{2r^2}\epsilon & \frac{\gamma}{2r}\delta-\frac{\alpha\beta}{2r^2}\epsilon \\
0 & 0 & -\frac{\beta}{2r}\delta+\frac{\alpha\gamma}{2r^2}\epsilon & \frac{\beta}{2r}\delta-\frac{\alpha\gamma}{2r^2}\epsilon & -\frac{\gamma}{2r}\delta-\frac{\alpha\beta}{2r^2}\epsilon & \frac{\gamma}{2r}\delta+\frac{\alpha\beta}{2r^2}\epsilon & 1-\frac{\beta^2+\gamma^2}{2r^2}\epsilon & \frac{\beta^2+\gamma^2}{2r^2}\epsilon \\
0 & 0 & \frac{\beta}{2r}\delta-\frac{\alpha\gamma}{2r^2}\epsilon & -\frac{\beta}{2r}\delta+\frac{\alpha\gamma}{2r^2}\epsilon & \frac{\gamma}{2r}\delta+\frac{\alpha\beta}{2r^2}\epsilon & -\frac{\gamma}{2r}\delta-\frac{\alpha\beta}{2r^2}\epsilon & \frac{\beta^2+\gamma^2}{2r^2}\epsilon & 1-\frac{\beta^2+\gamma^2}{2r^2}\epsilon
\end{pmatrix}$$

There are 5 eigenvalues of value 1, with constant eigenvectors. They are all sums of the members of the 5 distinct conjugacy classes. There is one additional eigenvalue of value 1, with variable eigenvector. The other two eigenvalues are $\lambda = e^{ir}$. Notice that the eigenvalues are functions only of r, so in order for the array V to be a function of α, β, and γ, the eigenvectors must be functions of α, β, and γ: in other words, not constant.

$$\begin{matrix} 1 \\ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{matrix} \quad \begin{matrix} 1 \\ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{matrix} \quad \begin{matrix} 1 \\ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{matrix} \quad \begin{matrix} 1 \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \end{matrix} \quad \begin{matrix} 1 \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \end{matrix} \quad \begin{matrix} 1 \\ \begin{pmatrix} 0 \\ 0 \\ \gamma \\ -\gamma \\ -\beta \\ \beta \\ \alpha \\ -\alpha \end{pmatrix} \end{matrix} \tag{38}$$

$$\begin{array}{cc} e^{ir} & e^{-ir} \\ \begin{pmatrix} 0 \\ 0 \\ \alpha^2+\beta^2 \\ -(\alpha^2+\beta^2) \\ (\beta\gamma+i\alpha r) \\ -(\beta\gamma+i\alpha r) \\ (-\alpha\gamma+i\beta r) \\ -(-\alpha\gamma+i\beta r) \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ \alpha^2+\beta^2 \\ -(\alpha^2+\beta^2) \\ (\beta\gamma-i\alpha r) \\ -(\beta\gamma-i\alpha r) \\ (-\alpha\gamma+i\beta r) \\ -(-\alpha\gamma-i\beta r) \end{pmatrix} \end{array} \tag{39}$$

Denote the eigenvectors in the order above by e_i. The first 5 are sums of members of the 5 distinct conjugacy classes, and commute with all the rest. Further, the product of any of e_2, e_3 or e_4 with any of e_5, e_6, or e_7 is zero. In addition, e_1 operating on any of the last three gives the negative of that eigenvector. The square of any of the last three is a factor times $e_1 - e_0$. The product of e_6 and e_7 is the complex conjugate of the product of e_7 and e_6. This leaves only the products of e_5 with e_6 and e_7, which do not commute.

Let us look further at the transformation, but restrict ourselves to $\beta = \gamma = 0$. Then $\alpha = r$, $\epsilon = 2\sin^2\alpha/2$, and $\delta = 2\sin\alpha/2\cos\alpha/2$. Let $\mu \equiv \cos^2\alpha/2$, $\nu \equiv \sin^2\alpha/2$, and $\omega \equiv \sin\alpha/2\cos\alpha/2$. Then the array reduces to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu & \nu & \omega & -\omega & 0 & 0 \\ 0 & 0 & \nu & \mu & -\omega & \omega & 0 & 0 \\ 0 & 0 & -\omega & \omega & \mu & \nu & 0 & 0 \\ 0 & 0 & \omega & -\omega & \nu & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

When $\mu = \cos^2\alpha/2 = 1$, then this becomes the identity. When $\nu = 1$, the inner array becomes

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where Λ swaps the elements c_2 and c_3 and the elements c_4 and c_5. This corresponds to eigenvalues of -1, and $r = \pi$. $\Lambda^2 = I$, of course. This suggests that the group displayed is SO(3).

There are permutations over the group elements possible here also. For unsigned permutations, there are 3 cases which turn out to be special cases of the above continous transform: ($2 \to 3$ and $4 \to 5$), ($2 \to 3$ and $6 \to 7$), and ($4 \to 5$ and $6 \to 7$). The first of these is illustrated directly above. There are two additional cases which are **not** cases of the original transformation from the identity–and thus may serve as bases for families of transformations themselves. These are ($2 \to 4$, $4 \to 6$, $6 \to 2$, with ($3 \to 5$, $5 \to 7$, $7 \to 3$), and ($2 \to 6$, $6 \to 4$, $4 \to 2$, with ($3 \to 7$, $7 \to 5$, $5 \to 3$). There are thus 3 families of transformations based on unsigned permutations of the group elements.

In addition we can consistently map *2* and *3* to -*2* and -*3*, provided we do the same with either the pair *4* and *5* or *6* and *7*. Likewise *4* and *5* can be mapped to their negatives if we do the same with *6* and *7*, giving us 4 possible sign mappings: the identity and 3 with 4 elements swapping sign. We have 12 families of transformations.

5 An Example with With a 10-Element Group

The non-abelian 10-element group may be defined by the following equations:

$$a^2 \to 0 \quad , \quad b^5 \to 0 \quad , \quad ab \to b^4 a \tag{40}$$

We may designate the elements of the group by *0* , *1* , *2* , *3* , *4* , *5* , *6* , *7*, *8*, and *9* , where *0* is the identity, $b \equiv \mathit{1}$, $b^2 \equiv \mathit{2}$, $b^3 \equiv \mathit{3}$, $b^4 \equiv \mathit{4}$, $a \equiv \mathit{5}$, $ab \equiv \mathit{6}$, $ab^2 \equiv \mathit{7}$, $ab^3 \equiv \mathit{8}$, and $ab^4 \equiv \mathit{9}$. We may define a product (Cayley) table for this group in the following way:

0	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
1	*2*	*3*	*4*	*0*	*9*	*5*	*6*	*7*	*8*
2	*3*	*4*	*0*	*1*	*8*	*9*	*5*	*6*	*7*
3	*4*	*0*	*1*	*2*	*7*	*8*	*9*	*5*	*6*
4	*0*	*1*	*2*	*3*	*6*	*7*	*8*	*9*	*5*
5	*6*	*7*	*8*	*9*	*0*	*1*	*2*	*3*	*4*
6	*7*	*8*	*9*	*5*	*4*	*0*	*1*	*2*	*3*
7	*8*	*9*	*5*	*6*	*3*	*4*	*0*	*1*	*2*
8	*9*	*5*	*6*	*7*	*2*	*3*	*4*	*0*	*1*
9	*5*	*6*	*7*	*8*	*1*	*2*	*3*	*4*	*0*

Grinding through the algebra will show that all elements of δV are zero except a few which are described by six independent tiny changes α, β, γ, ϵ, μ, and ν.

$$\delta V = \left(\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \nu & -\nu-\epsilon-\gamma & \nu+\epsilon+\gamma & -\nu \\ 0 & \mu & \nu+\epsilon & -\nu-\epsilon & -\mu \\ 0 & \epsilon & \mu+\gamma & -\mu-\gamma & -\epsilon \\ 0 & \gamma & -\nu-\mu-\gamma & \nu+\mu+\gamma & -\gamma \\ 0 & -\nu-\mu-\epsilon-\gamma & \nu+\gamma & -\nu-\gamma & \nu+\mu+\epsilon+\gamma \end{array} \right. \quad \cdots$$

$$\cdots \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ \nu & \mu & \epsilon & \gamma & -\nu-\mu-\gamma-\epsilon \\ -\nu-\epsilon-\gamma & \nu+\epsilon & \mu+\gamma & -\nu-\mu-\gamma & \nu+\gamma \\ \nu+\epsilon+\gamma & -\nu-\epsilon & -\mu-\gamma & \nu+\mu+\gamma & -\nu-\gamma \\ -\nu & -\mu & -\epsilon & -\gamma & \nu+\mu+\gamma+\epsilon \\ 0 & \alpha & \beta & -\beta & -\alpha \\ -\alpha & 0 & \alpha & \beta & -\beta \\ -\beta & -\alpha & 0 & \alpha & \beta \\ \beta & -\beta & -\alpha & 0 & \alpha \\ \alpha & \beta & -\beta & -\alpha & 0 \end{array} \right)$$

If we set

$$K = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \end{pmatrix} \quad , \quad L = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix} \quad , \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & -1 & 1 \\ 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 & 0 \\ 0 & L \end{pmatrix} \quad , \quad C = \begin{pmatrix} 0 & M \\ M^T & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & N \\ N^T & 0 \end{pmatrix} \quad , \quad E = \begin{pmatrix} 0 & P \\ P^T & 0 \end{pmatrix} \quad , \quad F = \begin{pmatrix} 0 & Q \\ Q^T & 0 \end{pmatrix}$$

The commutators of these generators are:

$$\begin{array}{r} [A,B]=0 \quad [A,C]=C-D-F \quad [A,D]=2C-E-F \\ [A,E]=C+D-2F \quad [A,F]=C+E-F \quad [B,C]=D-2E+F \\ [B,D]=D-E-F \quad [B,E]=C+D-E \quad [B,F]=-C+2D-E \\ [C,D]=2A-2B \quad [C,E]=2B \quad [C,F]=2*A-2B \\ [D,E]=2B \quad [D,F]=2B \quad [E,F]=2A-2B \end{array}$$

The eigenvalues of δV are given by the zeros of the following mess:

$$0 = x^6 \left(\begin{array}{ccccc}
 & -X^4 & & & \\
+X^2 (& 16\gamma\nu & +8\gamma\epsilon & +8\mu^2 & +8\mu\nu \\
+4\mu\epsilon & +12\nu^2 & +12\nu\epsilon & +8\epsilon^2 & -5\alpha^2 \\
-5\beta^2 & +12\gamma^2 & +\ 12\gamma\mu) & & \\
 & & & & \\
+(\ -112\gamma\nu\epsilon^2 & -32\gamma\epsilon^3 & -16\mu^4 & -32\mu^3\nu & -16\mu^3\epsilon \\
-64\mu^2\nu^2 & -64\mu^2\nu\epsilon & -16\mu^2\epsilon^2 & -48\mu\nu^3 & -112\mu\nu^2\epsilon \\
-96\mu\nu\epsilon^2 & -16\mu\epsilon^3 & -16\nu^4 & -32\nu^3\epsilon & -64\nu^2\epsilon^2 \\
-48\nu\epsilon^3 & -16\epsilon^4 & -5\alpha^4 & +10\alpha^3\beta & +5\alpha^2\beta^2 \\
+40\alpha^2\gamma^2 & +40\alpha^2\gamma\mu & +60\alpha^2\gamma\nu & +20\alpha^2\gamma\epsilon & +20\alpha^2\mu^2 \\
+20\alpha^2\mu\nu & +40\alpha^2\nu^2 & +40\alpha^2\nu\epsilon & +20\alpha^2\epsilon^2 & -10\alpha\beta^3 \\
+40\alpha\beta\gamma^2 & +40\alpha\beta\gamma\mu & +80\alpha\beta\gamma\nu & -40\alpha\beta\mu\epsilon & +40\alpha\beta\nu^2 \\
+40\alpha\beta\nu\epsilon & -5\beta^4 & +20\beta^2\gamma^2 & +20\beta^2\gamma\mu & +20\beta^2\gamma\nu \\
+20\beta^2\gamma\epsilon & +20\beta^2\mu^2 & +20\beta^2\mu\nu & +20\beta^2\mu\epsilon & +20\beta^2\nu^2 \\
+20\beta^2\nu\epsilon & +20\beta^2\epsilon^2 & -16\gamma^4 & -32\gamma^3\mu & -16\gamma^3\nu \\
-48\gamma^3\epsilon & -64\gamma^2\mu^2 & -64\gamma^2\mu\nu & -112\gamma^2\mu\epsilon & -16\gamma^2\nu^2 \\
-96\gamma^2\nu\epsilon & -64\gamma^2\epsilon^2 & -48\gamma\mu^3 & -112\gamma\mu^2\nu & -96\gamma\mu^2\epsilon \\
-96\gamma\mu\nu^2 & -176\gamma\mu\nu\epsilon & -64\gamma\mu\epsilon^2 & -16\gamma\nu^3 & -64\gamma\nu^2\epsilon\ )
\end{array} \right) \quad (41)$$

If all variables but μ are zero, then this has real eigenvalues, but if all but α are zero, then this has four imaginary eigenvalues.

6 A4

The alternating group of order 4 (the rotational symmetries of a tetrahedron) is a rather interesting group. It's Cayley table is

0	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
1	*0*	*7*	*11*	*6*	*9*	*4*	*2*	*10*	*5*	*8*	*3*
2	*5*	*6*	*10*	*8*	*11*	*0*	*3*	*9*	*4*	*7*	*1*
3	*4*	*8*	*9*	*7*	*10*	*5*	*1*	*11*	*0*	*6*	*2*
4	*3*	*1*	*2*	*5*	*0*	*7*	*8*	*6*	*10*	*11*	*9*
5	*2*	*3*	*1*	*0*	*4*	*8*	*6*	*7*	*11*	*9*	*10*
6	*11*	*0*	*7*	*9*	*1*	*2*	*10*	*4*	*8*	*3*	*5*
7	*9*	*4*	*8*	*10*	*3*	*1*	*11*	*5*	*6*	*2*	*0*
8	*10*	*5*	*6*	*11*	*2*	*3*	*9*	*0*	*7*	*1*	*4*
9	*7*	*11*	*0*	*1*	*6*	*10*	*4*	*2*	*3*	*5*	*8*
10	*8*	*9*	*4*	*3*	*7*	*11*	*5*	*1*	*2*	*0*	*6*
11	*6*	*10*	*5*	*2*	*8*	*9*	*0*	*3*	*1*	*4*	*7*

After grinding through the algebra one finds for it's differential matrix:

$$
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & \tau & \nu & \epsilon & -\beta & \ldots \\
0 & \nu & 0 & \tau & \delta-\tau & \gamma & \ldots \\
0 & \tau & \nu & 0 & -\alpha & \epsilon+\mu & \ldots \\
0 & \beta & \mu & -\alpha & 0 & 0 & \ldots \\
0 & -\epsilon & \gamma & \beta+\delta-\tau & 0 & 0 & \ldots \\
0 & -\tau & 0 & \alpha+\gamma & -\gamma & -\mu & \ldots \\
0 & \epsilon & \alpha & -\beta-\delta & \beta & -\gamma-\alpha & \ldots \\
0 & 0 & -\delta & \epsilon+\mu & -\epsilon-\mu & \beta+\delta-\tau & \ldots \\
0 & -\nu & -\gamma-\alpha & 0 & \tau-\beta-\delta & \alpha & \ldots \\
0 & 0 & \delta-\tau & -\epsilon-\mu-\nu & \mu & \tau-\delta & \ldots \\
0 & -\beta & -\mu-\nu & -\gamma & \gamma+\alpha & -\epsilon & \ldots
\end{array}
$$

$$
\begin{array}{ccccccc}
\ldots & 0 & 0 & 0 & 0 & 0 & 0 \\
\ldots & -\nu & \beta & 0 & -\tau & 0 & -\epsilon \\
\ldots & 0 & \alpha & -\nu-\mu & -\alpha-\gamma & \mu & -\delta \\
\ldots & \alpha+\gamma & -\epsilon-\nu-\mu & \beta+\delta-\tau & 0 & -\beta-\delta & -\gamma \\
\ldots & -\gamma & \epsilon & \tau-\beta-\delta & -\epsilon-\mu & \delta-\tau & \alpha+\gamma \\
\ldots & \tau-\delta & -\alpha-\gamma & \epsilon+\mu & \alpha & -\mu & -\beta \\
\ldots & 0 & \mu+\nu & \delta & -\nu & \tau-\delta & -\alpha \\
\ldots & \delta & 0 & \mu+\nu & \gamma & -\epsilon-\mu-\nu & 0 \\
\ldots & \mu+\nu & \delta & 0 & \tau-\beta-\delta & 0 & -\mu-\nu \\
\ldots & -\tau & \gamma & -\epsilon-\mu & 0 & \epsilon+\mu+\nu & \beta+\delta \\
\ldots & -\mu & -\beta-\delta & 0 & \beta+\delta & 0 & \epsilon+\mu+\nu \\
\ldots & -\alpha & 0 & -\delta & \epsilon+\mu+\nu & \beta+\delta & 0
\end{array}
$$

There are 8 generators that appear from the above, whose commutation relations look like

$$
\begin{array}{cc}
[A,B] = C-2F-H & [A,C] = B-2E-G \\
[A,D] = 0 & [A,E] = -C-H \\
[A,F] = -B-G & [A,G] = C-2F-H \\
[A,H] = B-2E-G & [B,C] = 0 \\
[B,D] = -C+F+2H & [B,E] = C-F-2H \\
[B,F] = 4A-B-2D+E+2G & [B,G] = -C+2F+H \\
[B,H] = -2A+B+4D-2E-G & [C,D] = -B+E+2G \\
[C,E] = -4A-B+2D+E+2G & [C,F] = C-F-2H \\
[C,G] = 2A+B-4D-2E-G & [C,H] = -C+2F+H \\
[D,E] = C-F-2H & [D,F] = B-E-2G \\
[D,G] = -C-F & [D,H] = -B-E \\
[E,F] = 0 & [E,G] = -2C+F+H \\
[E,H] = 2A+2B+2D-E-G & [F,G] = -2A+2B-2D-E-G \\
[F,H] = -2C+F+H & [G,H] = 0
\end{array}
$$

This is certainly messy, but some simplifications help.

7 Examples With 16-Element Groups

7.1 16-element group 1

One of the non-abelian 16-element groups may be defined as in the following table with the group elements designated by *0*, *1*, *2* . . . *15*. It generated by the relations $a^2 = 0$, $b^2 = a$, $c^2 = b^{-1}$, $d^2 = 0$, $a = b^{-1}d^{-1}bd$, and $b = c^{-1}d^{-1}cd$; where a is *4*, b is *2*, c is *3*, and d is *8*. The resulting product (Cayley) table is:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	2	3	4	5	6	7	0	14	15	13	12	8	9	10	11
2	3	4	5	6	7	0	1	10	11	9	8	14	15	13	12
3	4	5	6	7	0	1	2	13	12	15	14	10	11	9	8
4	5	6	7	0	1	2	3	9	8	11	10	13	12	15	14
5	6	7	0	1	2	3	4	15	14	12	13	9	8	11	10
6	7	0	1	2	3	4	5	11	10	8	9	15	14	12	13
7	0	1	2	3	4	5	6	12	13	14	15	11	10	8	9
8	12	11	15	9	13	10	14	0	4	6	2	1	5	7	3
9	13	10	14	8	12	11	15	4	0	2	6	5	1	3	7
10	14	8	12	11	15	9	13	2	6	0	4	3	7	1	5
11	15	9	13	10	14	8	12	6	2	4	0	7	3	5	1
12	11	15	9	13	10	14	8	7	3	5	1	0	4	6	2
13	10	14	8	12	11	15	9	3	7	1	5	4	0	2	6
14	8	12	11	15	9	13	10	1	5	7	3	2	6	0	4
15	9	13	10	14	8	12	11	5	1	3	7	6	2	4	0

This group has a set of infinitesimal transformations given by

$$\delta V = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda & \mu-\epsilon & -\rho & 0 & \rho & \epsilon-\mu & -\lambda \\
0 & \rho & \epsilon-\mu & -\lambda & 0 & \lambda & \mu-\epsilon & -\rho \\
0 & -\sigma & \nu-\epsilon & \lambda+\rho-\sigma & 0 & \sigma-\rho-\lambda & \epsilon-\nu & \sigma \\
0 & \sigma-\rho-\lambda & \epsilon-\nu & \sigma & 0 & -\sigma & \nu-\epsilon & \lambda+\rho-\sigma \\
0 & \mu & \sigma-\rho & -\nu & 0 & \nu & \rho-\sigma & -\mu \\
0 & \nu & \rho-\sigma & -\mu & 0 & \mu & \sigma-\rho & -\nu \\
0 & -\epsilon & \lambda-\sigma & \nu+\mu-\epsilon & 0 & \epsilon-\nu-\mu & \sigma-\lambda & \epsilon \\
0 & \epsilon-\mu-\nu & \sigma-\lambda & \epsilon & 0 & -\epsilon & \lambda-\sigma & \mu+\nu-\epsilon
\end{pmatrix} \cdots$$

$$
\cdots \left.\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\lambda & \rho & -\sigma & \sigma-\rho-\lambda & \mu & \nu & -\epsilon & \epsilon-\mu-\nu \\
\mu-\epsilon & \epsilon-\mu & \nu-\epsilon & \epsilon-\nu & \sigma-\rho & \rho-\sigma & \lambda-\sigma & \sigma-\lambda \\
-\rho & -\lambda & \rho+\lambda-\sigma & \sigma & -\nu & -\mu & \mu+\nu-\epsilon & \epsilon \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\rho & \lambda & \sigma-\rho-\lambda & -\sigma & \nu & \mu & \epsilon-\mu-\nu & -\epsilon \\
\epsilon-\mu & \mu-\epsilon & \epsilon-\nu & \nu-\epsilon & \rho-\sigma & \sigma-\rho & \sigma-\lambda & \lambda-\sigma \\
-\lambda & -\rho & \sigma & \lambda+\rho-\sigma & -\mu & -\nu & \epsilon & \mu+\nu-\epsilon \\
0 & 0 & -\alpha & \alpha & -\beta & -\gamma & \beta & \gamma \\
0 & 0 & \alpha & -\alpha & -\gamma & -\beta & \gamma & \beta \\
\alpha & -\alpha & 0 & 0 & \gamma & \beta & -\beta & -\gamma \\
-\alpha & \alpha & 0 & 0 & \beta & \gamma & -\gamma & -\beta \\
\beta & \gamma & -\gamma & -\beta & 0 & 0 & -\alpha & \alpha \\
\gamma & \beta & -\beta & -\gamma & 0 & 0 & \alpha & -\alpha \\
-\beta & -\gamma & \beta & \gamma & \alpha & -\alpha & 0 & 0 \\
-\gamma & -\beta & \gamma & \beta & -\alpha & \alpha & 0 & 0
\end{array}\right)
$$

Here δV is given by $\alpha A' + \beta B' + \gamma G' + \mu M' + \nu N' + \rho R' + \sigma S' + \lambda L' + \epsilon E'$. If we combine these we can simplify the commutation relations. For instance, let $E = M' + N' + E'$, $L = R' + S' + L'$, and $G = G' + B'$. Then the 36 non-trivial commutation relations appear as

$$
\begin{array}{rcl}
[A,B]=0 & [A,G]=0 & [A,L]=0 \\
[A,E]=0 & [B,G]=0 & [G,S]=0 \\
[M,E]=0 & [N,S]=0 & [N,E]=0 \\
[R,L]=0 & [S,L]=0 & [S,E]=0 \\
[A,M]=E-2N & [A,N]=2M-E & [A,R]=L2R-2S \\
[A,S]=4R+2S-2L & [B,M]=L-S & [B,N]=2R+S \\
[B,R]=-M-N & [B,S]=2M-ED & [B,L]=-2E \\
[B,E]=2L & [G,M]=2L & [G,N]=2L \\
[G,R]=-2E & [G,L]=-4E & [G,E]=4L \\
[M,N]=2A & [M,R]=-2B & [M,S]=4B-2G \\
[M,L]=-2G & [N,R]=-2B & [N,L]=-2G \\
[R,S]=2A & [R,E]=2G & [L,E]=4G
\end{array}
$$

The generators A, B, and G form one subset. Note that the commutators of G, L, and E cycle, as do those of A, $E-2M$, and $E-2N$. At the moment I am unable to identify this with one of the classical Lie groups.

The eigenfunction of the differential tranformation array is rather messy, but has the form

$$0 = x^8 \left(x^6 + ()\, x^4 + ()\, x^2 + () \right)$$

where the last quantity in parenthesis is about 2 pages long. The quantities inside the parentheses are polynomials in the differential changes.

7.2 16-element group 2

Another one of the non-abelian 16-element groups may be defined as in the following table with the group elements designated by *0*, *1*, *2* . . . *15*. It generated by the relations $a^2 = b^2 = c^2 = 0$, $x^2 = a$, and $a = b^{-1}c^{-1}bc$, where x is *1*, a is *2*, b is *9*, and c is *13*. The resulting product (Cayley) table is:

0	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*
1	*2*	*3*	*0*	*5*	*6*	*7*	*4*	*9*	*10*	*11*	*8*	*13*	*14*	*15*	*12*
2	*3*	*0*	*1*	*6*	*7*	*4*	*5*	*10*	*11*	*8*	*9*	*14*	*15*	*12*	*13*
3	*0*	*1*	*2*	*7*	*4*	*5*	*6*	*11*	*8*	*9*	*10*	*15*	*12*	*13*	*14*
4	*5*	*6*	*7*	*2*	*3*	*0*	*1*	*14*	*15*	*12*	*13*	*8*	*9*	*10*	*11*
5	*6*	*7*	*4*	*3*	*0*	*1*	*2*	*15*	*12*	*13*	*14*	*9*	*10*	*11*	*8*
6	*7*	*4*	*5*	*0*	*1*	*2*	*3*	*12*	*13*	*14*	*15*	*10*	*11*	*8*	*9*
7	*4*	*5*	*6*	*1*	*2*	*3*	*0*	*13*	*14*	*15*	*12*	*11*	*8*	*9*	*10*
8	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*2*	*3*	*0*	*1*	*6*	*7*	*4*	*5*
9	*10*	*11*	*8*	*13*	*14*	*15*	*12*	*3*	*0*	*1*	*2*	*7*	*4*	*5*	*6*
10	*11*	*8*	*9*	*14*	*15*	*12*	*13*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
11	*8*	*9*	*10*	*15*	*12*	*13*	*14*	*1*	*2*	*3*	*0*	*5*	*6*	*7*	*4*
12	*13*	*14*	*15*	*10*	*11*	*8*	*9*	*4*	*5*	*6*	*7*	*2*	*3*	*0*	*1*
13	*14*	*15*	*12*	*11*	*8*	*9*	*10*	*5*	*6*	*7*	*4*	*3*	*0*	*1*	*2*
14	*15*	*12*	*13*	*8*	*9*	*10*	*11*	*6*	*7*	*4*	*5*	*0*	*1*	*2*	*3*
15	*12*	*13*	*14*	*9*	*10*	*11*	*8*	*7*	*4*	*5*	*6*	*1*	*2*	*3*	*0*

This group has a set of infinitesimal transformations given by

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	ϵ	μ	$-\epsilon$	$-\mu$	γ	$-\delta$	$-\gamma$	δ
0	0	0	0	0	0	0	0	$-\mu$	ϵ	μ	$-\epsilon$	δ	γ	$-\delta$	$-\gamma$
0	0	0	0	0	0	0	0	$-\epsilon$	$-\mu$	ϵ	μ	$-\gamma$	δ	γ	$-\delta$
0	0	0	0	0	0	0	0	μ	$-\epsilon$	$-\mu$	ϵ	$-\delta$	$-\gamma$	δ	γ
0	0	0	0	$-\epsilon$	$-\mu$	ϵ	μ	0	0	0	0	$-\alpha$	β	α	$-\beta$
0	0	0	0	μ	$-\epsilon$	$-\mu$	ϵ	0	0	0	0	$-\beta$	$-\alpha$	β	α
0	0	0	0	ϵ	μ	$-\epsilon$	$-\mu$	0	0	0	0	α	$-\beta$	$-\alpha$	β
0	0	0	0	$-\mu$	ϵ	μ	$-\epsilon$	0	0	0	0	β	α	$-\beta$	$-\alpha$
0	0	0	0	$-\gamma$	δ	γ	$-\delta$	α	$-\beta$	$-\alpha$	β	0	0	0	0
0	0	0	0	$-\delta$	$-\gamma$	δ	γ	β	α	$-\beta$	$-\alpha$	0	0	0	0
0	0	0	0	γ	$-\delta$	$-\gamma$	δ	$-\alpha$	β	α	$-\beta$	0	0	0	0
0	0	0	0	δ	γ	$-\delta$	$-\gamma$	$-\beta$	$-\alpha$	β	α	0	0	0	0

If we define S and T such that

$$S \equiv \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \quad T \equiv \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -S \\ 0 & 0 & S & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & T \\ 0 & 0 & -T & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & S \\ 0 & 0 & 0 & 0 \\ 0 & -S & 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & T \\ 0 & 0 & 0 & 0 \\ 0 & -T & 0 & 0 \end{pmatrix}$$

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & S & 0 \\ 0 & -S & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & T & 0 \\ 0 & -T & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Notice that $S^2 = S$, $T^2 = -S$, and $ST = TS = T$. These two arrays are thus isomorphic to 1 and i. The commutation relations among the generators are:

$$\begin{aligned} [A,B] = 0 \quad [C,D] = 0 \quad [E,F] = 0 \\ [C,E] = -A \quad [E,A] = -C \quad [A,C] = -E \\ [D,F] = A \quad [F,A] = -D \quad [A,D] = -F \\ [D,E] = B \quad [E,B] = D \quad [B,D] = -E \\ [C,F] = B \quad [F,B] = -C \quad [B,C] = F \end{aligned}$$

These are the generators of the special group Sl(2,c), the set of 2x2 complex matrices with determinant 1? This is NOT isomorphic to the previous group.

7.3 16-element group 3

One of the non-abelian 16-element groups may be defined as in the following table with the group elements designated by The relations $a^2 = b^2 = x$, $x = a^{-1}b^{-1}ab$, and $x^2 = y^2 = 0$ define a non-abelian 16-element group, with group elements I will designate *0*, *1*, *2* . . . *15*, where a is *8*, b is *12*, x is *1*, and y is *2*. This gives the following product (Cayley) table:

$$
\begin{array}{cccccccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 & 9 & 8 & 11 & 10 & 13 & 12 & 15 & 14 \\
2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 & 10 & 11 & 8 & 9 & 14 & 15 & 12 & 13 \\
3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 & 11 & 10 & 9 & 8 & 15 & 14 & 13 & 12 \\
4 & 5 & 6 & 7 & 1 & 0 & 3 & 2 & 12 & 13 & 14 & 15 & 9 & 8 & 11 & 10 \\
5 & 4 & 7 & 6 & 0 & 1 & 2 & 3 & 13 & 12 & 15 & 14 & 8 & 9 & 10 & 11 \\
6 & 7 & 4 & 5 & 3 & 2 & 1 & 0 & 14 & 15 & 12 & 13 & 11 & 10 & 9 & 8 \\
7 & 6 & 5 & 4 & 2 & 3 & 0 & 1 & 15 & 14 & 13 & 12 & 10 & 11 & 8 & 9 \\
8 & 9 & 10 & 11 & 13 & 12 & 15 & 14 & 1 & 0 & 3 & 2 & 4 & 5 & 6 & 7 \\
9 & 8 & 11 & 10 & 12 & 13 & 14 & 15 & 0 & 1 & 2 & 3 & 5 & 4 & 7 & 6 \\
10 & 11 & 8 & 9 & 15 & 14 & 13 & 12 & 3 & 2 & 1 & 0 & 6 & 7 & 4 & 5 \\
11 & 10 & 9 & 8 & 14 & 15 & 12 & 13 & 2 & 3 & 0 & 1 & 7 & 6 & 5 & 4 \\
12 & 13 & 14 & 15 & 8 & 9 & 10 & 11 & 5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 \\
13 & 12 & 15 & 14 & 9 & 8 & 11 & 10 & 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\
14 & 15 & 12 & 13 & 10 & 11 & 8 & 9 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\
15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1
\end{array}
\tag{42}
$$

Using the standard techniques, we find that δ V depends on 6 parameters as follows:

$$
\delta V = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon & -\epsilon & \mu & -\mu & \gamma & -\gamma & \delta & -\delta \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\epsilon & \epsilon & -\mu & \mu & -\gamma & \gamma & -\delta & \delta \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & -\mu & \epsilon & -\epsilon & \delta & -\delta & \gamma & -\gamma \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & \mu & -\epsilon & \epsilon & -\delta & \delta & -\gamma & \gamma \\
0 & 0 & 0 & 0 & -\epsilon & \epsilon & -\mu & \mu & 0 & 0 & 0 & 0 & \alpha & -\alpha & \beta & -\beta \\
0 & 0 & 0 & 0 & \epsilon & -\epsilon & \mu & -\mu & 0 & 0 & 0 & 0 & -\alpha & \alpha & -\beta & \beta \\
0 & 0 & 0 & 0 & -\mu & \mu & -\epsilon & \epsilon & 0 & 0 & 0 & 0 & \beta & -\beta & \alpha & -\alpha \\
0 & 0 & 0 & 0 & \mu & -\mu & \epsilon & -\epsilon & 0 & 0 & 0 & 0 & -\beta & \beta & -\alpha & \alpha \\
0 & 0 & 0 & 0 & -\gamma & \gamma & -\delta & \delta & -\alpha & \alpha & -\beta & \beta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \gamma & -\gamma & \delta & -\delta & \alpha & -\alpha & \beta & -\beta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\delta & \delta & -\gamma & \gamma & -\beta & \beta & -\alpha & \alpha & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \delta & -\delta & \gamma & -\gamma & \beta & -\beta & \alpha & -\alpha & 0 & 0 & 0 & 0
\end{pmatrix}
\tag{43}
$$

A simple pattern is immediately evident. Let

$$
Q \equiv \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad R = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} \quad S = \begin{pmatrix} 0 & Q \\ Q & 0 \end{pmatrix}
\tag{44}
$$

It is fairly easy to see that $RS = SR = -S$ and that $R^2 = S^2 = -R$. Now define the 'infinitesimal generators' in our space, A through F as

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -R \\ 0 & 0 & R & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -S \\ 0 & 0 & S & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -R \\ 0 & 0 & 0 & 0 \\ 0 & R & 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -S \\ 0 & 0 & 0 & 0 \\ 0 & S & 0 & 0 \end{pmatrix}$$

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -R & 0 \\ 0 & R & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -S & 0 \\ 0 & S & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The commutators of these generators are very simple:

$$\begin{array}{rrr} [A,B] = 0 & [C,D] = 0 & [E,F] = 0 \\ [C,E] = A & [E,A] = C & [A,C] = E \\ [D,F] = A & [F,A] = D & [A,D] = F \\ [C,F] = B & [F,B] = C & [B,C] = F \\ [D,E] = B & [E,B] = D & [B,D] = E \end{array}$$

Using the notation of Gilmore [2], if I make the identifications

$$A = O_{13} \quad B = O_{24} \quad C = O_{12} \quad D = -O_{34} \quad E = O_{23} \quad F = -O_{14}$$

these are just the generators of the group SO(4). This is NOT isomorphic to the previous group.

7.4 16-element group 4

Another of the 16-element groups is generated by the elements $3^2 = 11$, $8^2 = 0$, $11 = 3^{-1}8^{-1}38$, and $11^2 = 4^2 = 0$, with $11x = x11$ and $4x = x4$.

[2] Lie Groups, Lie Algebras, and Some of Their Applications, page 187

0	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*
1	*0*	*3*	*2*	*5*	*4*	*7*	*6*	*9*	*8*	*11*	*10*	*13*	*12*	*15*	*14*
2	*3*	*0*	*1*	*6*	*7*	*4*	*5*	*14*	*15*	*12*	*13*	*10*	*11*	*8*	*9*
3	*2*	*1*	*0*	*7*	*6*	*5*	*4*	*15*	*14*	*13*	*12*	*11*	*10*	*9*	*8*
4	*5*	*6*	*7*	*1*	*0*	*3*	*2*	*11*	*10*	*8*	*9*	*14*	*15*	*13*	*12*
5	*4*	*7*	*6*	*0*	*1*	*2*	*3*	*10*	*11*	*9*	*8*	*15*	*14*	*12*	*13*
6	*7*	*4*	*5*	*3*	*2*	*1*	*0*	*13*	*12*	*14*	*15*	*8*	*9*	*11*	*10*
7	*6*	*5*	*4*	*2*	*3*	*0*	*1*	*12*	*13*	*15*	*14*	*9*	*8*	*10*	*11*
8	*9*	*14*	*15*	*10*	*11*	*12*	*13*	*0*	*1*	*4*	*5*	*6*	*7*	*2*	*3*
9	*8*	*15*	*14*	*11*	*10*	*13*	*12*	*1*	*0*	*5*	*4*	*7*	*6*	*3*	*2*
10	*11*	*12*	*13*	*9*	*8*	*15*	*14*	*5*	*4*	*0*	*1*	*2*	*3*	*7*	*6*
11	*10*	*13*	*12*	*8*	*9*	*14*	*15*	*4*	*5*	*1*	*0*	*3*	*2*	*6*	*7*
12	*13*	*10*	*11*	*15*	*14*	*9*	*8*	*7*	*6*	*2*	*3*	*0*	*1*	*5*	*4*
13	*12*	*11*	*10*	*14*	*15*	*8*	*9*	*6*	*7*	*3*	*2*	*1*	*0*	*4*	*5*
14	*15*	*8*	*9*	*12*	*13*	*10*	*11*	*2*	*3*	*6*	*7*	*4*	*5*	*0*	*1*
15	*14*	*9*	*8*	*13*	*12*	*11*	*10*	*3*	*2*	*7*	*6*	*5*	*4*	*1*	*0*

The δV for this has 6 variables: α, β, γ, ϵ, μ, and σ; which multiply arrays I denote by A, B, C, D, E, and F respectively.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	γ	$-\gamma$	μ	$-\mu$	σ	$-\sigma$	ϵ	$-\epsilon$
0	0	0	0	0	0	0	0	$-\gamma$	γ	$-\mu$	μ	$-\sigma$	σ	$-\epsilon$	ϵ
0	0	0	0	0	0	0	0	ϵ	$-\epsilon$	σ	$-\sigma$	μ	$-\mu$	γ	$-\gamma$
0	0	0	0	0	0	0	0	$-\epsilon$	ϵ	$-\sigma$	σ	$-\mu$	μ	$-\gamma$	γ
0	0	0	0	γ	$-\gamma$	ϵ	$-\epsilon$	0	0	α	$-\alpha$	β	$-\beta$	0	0
0	0	0	0	$-\gamma$	γ	$-\epsilon$	ϵ	0	0	$-\alpha$	α	$-\beta$	β	0	0
0	0	0	0	μ	$-\mu$	σ	$-\sigma$	$-\alpha$	α	0	0	0	0	$-\beta$	β
0	0	0	0	$-\mu$	μ	$-\sigma$	σ	α	$-\alpha$	0	0	0	0	β	$-\beta$
0	0	0	0	σ	$-\sigma$	μ	$-\mu$	$-\beta$	β	0	0	0	0	$-\alpha$	α
0	0	0	0	$-\sigma$	σ	$-\mu$	μ	β	$-\beta$	0	0	0	0	α	$-\alpha$
0	0	0	0	ϵ	$-\epsilon$	γ	$-\gamma$	0	0	β	$-\beta$	α	$-\alpha$	0	0
0	0	0	0	$-\epsilon$	ϵ	$-\gamma$	γ	0	0	$-\beta$	β	$-\alpha$	α	0	0

The eigenvalues are given by the solutions to

$$0 = X^{12}\left(\begin{array}{c} X^4 \\ +8X^2\left(\alpha^2+\beta^2-\epsilon^2-\gamma^2-\mu^2-\sigma^2\right) \\ +32\left(\begin{array}{ccccc} \epsilon^2\mu^2 & +\epsilon^2\sigma^2 & +\gamma^2\mu^2 & +\gamma^2\sigma^2 & \alpha^2\beta^2 \\ -\alpha^2\epsilon^2 & -\alpha^2\gamma^2 & -\alpha^2\mu^2 & -\alpha^2\sigma^2 & -\beta^2\epsilon^2 \\ -\beta^2\gamma^2 & -\beta^2\mu^2 & -\beta^2\sigma^2 & -\mu^2\sigma^2 & -\epsilon^2\gamma^2 \end{array}\right) \\ +128\alpha\beta\epsilon\gamma+128\alpha\beta\mu\sigma-128\epsilon\gamma\mu\sigma \\ +16\alpha^4+16\beta^4+16\epsilon^4+16\gamma^4+16\mu^4+16\sigma^4 \end{array}\right)$$

The generator commutation relations are as follows:

$$\begin{array}{ccc} [A,B]=0 & [C,D]=0 & [E,F]=0 \\ [C,E]=A & [E,A]=-C & [A,C]=-E \\ [C,F]=B & [F,B]=-C & [B,C]=-F \\ [D,E]=B & [E,B]=-D & [B,D]=-E \\ [D,F]=A & [F,A]=-D & [A,D]=-F \end{array}$$

7.5 16-element group 5

Another 16-element group is generated by $1^2 = 0$, $2^2 = 4$, $4^2 = 8^2 = 0$, $8 = 1^{-1}2^{-1}12$, $8x = x8$, and $4x = x4$. The product (Cayley) table here is

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	11	4	13	6	15	0	9	10	3	12	5	14	7	8	1
3	10	5	12	7	14	1	8	11	2	13	4	15	6	9	0
4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
6	15	0	9	2	11	4	13	14	7	8	1	10	3	12	5
7	14	1	8	3	10	5	12	15	6	9	0	11	2	13	4
8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	3	12	5	14	7	8	1	2	11	4	13	6	15	0	9
11	2	13	4	15	6	9	0	3	10	5	12	7	14	1	8
12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
14	7	8	1	10	3	12	5	6	15	0	9	2	11	4	13
15	6	9	0	11	2	13	4	7	14	1	8	3	10	5	12

This results in a δV of the form

$$\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu & -\beta & 0 & 0 & -\sigma & -\epsilon & 0 & 0 & -\mu & \beta & 0 & 0 & \sigma & \epsilon \\
0 & \sigma & 0 & -\alpha & 0 & -\mu & 0 & -\gamma & 0 & -\sigma & 0 & \alpha & 0 & \mu & 0 & \gamma \\
0 & -\epsilon & -\alpha & 0 & 0 & -\beta & -\gamma & 0 & 0 & \epsilon & \alpha & 0 & 0 & \beta & \gamma & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\sigma & -\epsilon & 0 & 0 & \mu & -\beta & 0 & 0 & \sigma & \epsilon & 0 & 0 & -\mu & \beta \\
0 & -\mu & 0 & -\gamma & 0 & \sigma & 0 & -\alpha & 0 & \mu & 0 & \gamma & 0 & -\sigma & 0 & \alpha \\
0 & -\beta & -\gamma & 0 & 0 & -\epsilon & -\alpha & 0 & 0 & \beta & \gamma & 0 & 0 & \epsilon & \alpha & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\mu & \beta & 0 & 0 & \sigma & \epsilon & 0 & 0 & \mu & -\beta & 0 & 0 & -\sigma & -\epsilon \\
0 & -\sigma & 0 & \alpha & 0 & \mu & 0 & \gamma & 0 & \sigma & 0 & -\alpha & 0 & -\mu & 0 & -\gamma \\
0 & \epsilon & \alpha & 0 & 0 & \beta & \gamma & 0 & 0 & -\epsilon & -\alpha & 0 & 0 & -\beta & -\gamma & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sigma & \epsilon & 0 & 0 & -\mu & \beta & 0 & 0 & -\sigma & -\epsilon & 0 & 0 & \mu & -\beta \\
0 & \mu & 0 & \gamma & 0 & -\sigma & 0 & \alpha & 0 & -\mu & 0 & -\gamma & 0 & \sigma & 0 & -\alpha \\
0 & \beta & \gamma & 0 & 0 & \epsilon & \alpha & 0 & 0 & -\beta & -\gamma & 0 & 0 & -\epsilon & -\alpha & 0
\end{matrix}$$

The commutation relations among the generators are as follows, where $2\alpha A + 2\beta B + 2\gamma C + 2\epsilon D + 2\mu E + 2\sigma F = \delta V$:

$$\begin{matrix}
[A,C]=0 & [B,D]=A & [B,F]=-C & [E,D]=C & [E,F]=-A \\
[B,E]=0 & [D,A]=E & [F,C]=-E & [D,C]=B & [F,A]=-B \\
[D,F]=0 & [A,E]=-D & [C,E]=F & [C,B]=-D & [A,B]=F
\end{matrix}$$

The eigenfunction for the differential array is given below. There are 10 constant eigenvectors with eigenvalue 0 (for the differential array) or 1 (for the full transformation.

$$0 = X^{12}\left(\begin{matrix}
X^4 \\
+X^2\left(-16\sigma\mu + \left(-16\epsilon\beta + \left(-8\alpha^2 - 8\gamma^2\right)\right)\right) \\
+ \\
-16\mu^4 + \left(32\beta^2 + \left(32\sigma^2 + \left(32\epsilon^2 + 64\gamma\alpha\right)\right)\right)\mu^2 \\
+\left(128\epsilon\sigma\beta + \left(64\alpha^2 + 64\gamma^2\right)\sigma\right)\mu \\
+\left(-16\beta^4 + \left(32\sigma^2 + \left(32\epsilon^2 - 64\gamma\alpha\right)\right)\beta^2\right. \\
+\left(64\alpha^2 + 64\gamma^2\right)\epsilon\beta + \left(-16\sigma^4 + \left(32\epsilon^2 + 64\gamma\alpha\right)\sigma^2\right. \\
\left.\left.+\left(-16\epsilon^4 - 64\gamma\alpha\epsilon^2 + \left(16\alpha^4 - 32\gamma^2\alpha^2 + 16\gamma^4\right)\right)\right)\right)
\end{matrix}\right)$$

7.6 16-element group 6

There is a 6'th 16-element non-abelian group, generated by the following relations: $\mathit{1}^2 = \mathit{4}$, $\mathit{2}^2 = \mathit{8}$, $\mathit{4} = \mathit{1}^{-1}\mathit{2}^{-1}\mathit{12}$, and $\mathit{4}^2 = \mathit{8}^2 = \mathit{0}$, with $\mathit{4}x = x\mathit{4}$, $\mathit{8}x = x\mathit{8}$. It's product (Cayley) table is given by

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	4	3	6	5	0	7	2	9	12	11	14	13	8	15	10
2	7	8	13	6	3	12	9	10	15	0	5	14	11	4	1
3	2	9	8	7	6	13	12	11	10	1	0	15	14	5	4
4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	0	7	2	1	4	3	6	13	8	15	10	9	12	11	14
6	3	12	9	2	7	8	13	14	11	4	1	10	15	0	5
7	6	13	12	3	2	9	8	15	14	5	4	11	10	1	0
8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	12	11	14	13	8	15	10	1	4	3	6	5	0	7	2
10	15	0	5	14	11	4	1	2	7	8	13	6	3	12	9
11	10	1	0	15	14	5	4	3	2	9	8	7	6	13	12
12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
13	8	15	10	9	12	11	14	5	0	7	2	1	4	3	6
14	11	4	1	10	15	0	5	6	3	12	9	2	7	8	13
15	14	5	4	11	10	1	0	7	6	13	12	3	2	9	8

Solving for the differential gives the following somewhat unusual array. Note that an element is no longer equal to the absolute value of the element in its transpose position.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	μ	$-\beta$	0	0	$-\mu$	β	0	0	$-\sigma$	$-\epsilon$	0	0	σ	ϵ
0	$-\sigma$	0	$-\alpha$	0	σ	0	α	0	μ	0	$-\gamma$	0	$-\mu$	0	γ
0	$-\epsilon$	α	0	0	ϵ	$-\alpha$	0	0	$-\beta$	γ	0	0	β	$-\gamma$	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	$-\mu$	β	0	0	μ	$-\beta$	0	0	σ	ϵ	0	0	$-\sigma$	$-\epsilon$
0	σ	0	α	0	$-\sigma$	0	$-\alpha$	0	$-\mu$	0	γ	0	μ	0	$-\gamma$
0	ϵ	$-\alpha$	0	0	$-\epsilon$	α	0	0	β	$-\gamma$	0	0	$-\beta$	γ	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	$-\sigma$	$-\epsilon$	0	0	σ	ϵ	0	0	μ	$-\beta$	0	0	$-\mu$	β
0	μ	0	$-\gamma$	0	$-\mu$	0	γ	0	$-\sigma$	0	$-\alpha$	0	σ	0	α
0	$-\beta$	γ	0	0	β	$-\gamma$	0	0	$-\epsilon$	α	0	0	ϵ	$-\alpha$	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	σ	ϵ	0	0	$-\sigma$	$-\epsilon$	0	0	$-\mu$	β	0	0	μ	$-\beta$
0	$-\mu$	0	γ	0	μ	0	$-\gamma$	0	σ	0	α	0	$-\sigma$	0	$-\alpha$
0	β	$-\gamma$	0	0	$-\beta$	γ	0	0	ϵ	$-\alpha$	0	0	$-\epsilon$	α	0

The eigenvalue of this differential transform are (found by REDUCE)

$$X^{12}\left(\begin{array}{ccccc} 16\alpha^4 & -16\sigma^4 & -16\mu^4 & +16\gamma^4 & -16\beta^4 \\ -16\epsilon^4 & -64\alpha^2\beta\epsilon & +32\mu^2\sigma^2 & -32\alpha^2\gamma^2 & +64\alpha^2\mu\sigma \\ +64\alpha\beta^2\gamma & +64\alpha\epsilon^2\gamma & +64\alpha\gamma\mu^2 & +64\alpha\gamma\sigma^2 & +32\beta^2\epsilon^2 \\ -32\beta^2\mu^2 & -32\beta^2\sigma^2 & -64\beta\epsilon\gamma^2 & -128\beta\epsilon\mu\sigma & -32\epsilon^2\mu^2 \\ -32\epsilon^2\sigma^2 & +64\gamma^2\mu\sigma & & & \\ -16\beta\epsilon X^2 & +8\gamma^2 X^2 & +16\mu\sigma X^2 & +8\alpha^2 X^2 & \\ +X^4 & & & & \end{array}\right)$$

Natural generators are given by $\delta V = 2\alpha A + 2\beta B + 2\gamma C + 2\epsilon D + 2\mu E + 2\sigma F$. The commutation relation among these generators are

$$\begin{array}{lllll} [A,C]=0 & [B,E]=A & [B,F]=-C & [D,E]=-C & [D,F]=A \\ [B,D]=0 & [E,A]=B & [F,C]=-B & [E,C]=-D & [F,A]=D \\ [E,F]=0 & [A,B]=-F & [C,B]=E & [C,D]=F & [A,D]=-E \end{array}$$

7.7 16-element group 7

The seventh of our non-abelian 16-element groups is generated by the relations $4^2 = 8^2 = 0$, $1^2 = 2$, $2^2 = 4$, and $4 = 8^{-1}1^{-1}81$.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	2	3	4	5	6	7	0	9	10	11	12	13	14	15	8
2	3	4	5	6	7	0	1	10	11	12	13	14	15	8	9
3	4	5	6	7	0	1	2	11	12	13	14	15	8	9	10
4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	6	7	0	1	2	3	4	13	14	15	8	9	10	11	12
6	7	0	1	2	3	4	5	14	15	8	9	10	11	12	13
7	0	1	2	3	4	5	6	15	8	9	10	11	12	13	14
8	13	10	15	12	9	14	11	0	5	2	7	4	1	6	3
9	14	11	8	13	10	15	12	1	6	3	0	5	2	7	4
10	15	12	9	14	11	8	13	2	7	4	1	6	3	0	5
11	8	13	10	15	12	9	14	3	0	5	2	7	4	1	6
12	9	14	11	8	13	10	15	4	1	6	3	0	5	2	7
13	10	15	12	9	14	11	8	5	2	7	4	1	6	3	0
14	11	8	13	10	15	12	9	6	3	0	5	2	7	4	1
15	12	9	14	11	8	13	10	7	4	1	6	3	0	5	2

The resulting δV is given by

$$
\begin{array}{cccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & -\gamma & -\sigma & -\epsilon & -\mu & \gamma & \sigma & \epsilon \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma & \epsilon & \mu & -\gamma & -\sigma & -\epsilon & -\mu & \gamma \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & \gamma & \sigma & \epsilon & \mu & -\gamma & -\sigma & -\epsilon \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sigma & -\epsilon & -\mu & \gamma & \sigma & \epsilon & \mu & -\gamma \\
0 & \sigma & 0 & \mu & 0 & -\sigma & 0 & -\mu & 0 & -\alpha & 0 & -\beta & 0 & \alpha & 0 & \beta \\
0 & -\gamma & 0 & -\epsilon & 0 & \gamma & 0 & \epsilon & \beta & 0 & -\alpha & 0 & -\beta & 0 & \alpha & 0 \\
0 & -\mu & 0 & \sigma & 0 & \mu & 0 & -\sigma & 0 & \beta & 0 & -\alpha & 0 & -\beta & 0 & \alpha \\
0 & \epsilon & 0 & -\gamma & 0 & -\epsilon & 0 & \gamma & \alpha & 0 & \beta & 0 & -\alpha & 0 & -\beta & 0 \\
0 & -\sigma & 0 & -\mu & 0 & \sigma & 0 & \mu & 0 & \alpha & 0 & \beta & 0 & -\alpha & 0 & -\beta \\
0 & \gamma & 0 & \epsilon & 0 & -\gamma & 0 & -\epsilon & -\beta & 0 & \alpha & 0 & \beta & 0 & -\alpha & 0 \\
0 & \mu & 0 & -\sigma & 0 & -\mu & 0 & \sigma & 0 & -\beta & 0 & \alpha & 0 & \beta & 0 & -\alpha \\
0 & -\epsilon & 0 & \gamma & 0 & \epsilon & 0 & -\gamma & -\alpha & 0 & -\beta & 0 & \alpha & 0 & \beta & 0
\end{array}
$$

Once again, the generators themselves are sometimes neither symmetric nor anti-symmetric. The natural generators are just the arrays corresponding to the above differentials, divided by 2. $\delta V = 2\alpha A + 2\beta B + 2\gamma C + 2\epsilon D + 2\mu E + 2\sigma F$. Their commutators are

$$
\begin{array}{ccccc}
[A,B]=0 & [C,D]=A & [E,D]=B & [C,F]=B & [E,F]=-A \\
[C,E]=0 & [D,A]=E & [D,B]=-C & [F,B]=-E & [F,A]=-C \\
[D,F]=0 & [A,E]=-F & [B,C]=-F & [B,E]=D & [A,C]=-D
\end{array}
$$

The eigenvalues of this differential transform are the solutions to the following equation (found by REDUCE):

$$
X^{12}\left(\begin{array}{ccccc}
16\alpha^4 & +32\alpha^2\beta^2 & +64\alpha^2\epsilon\gamma & +32\alpha^2\mu^2 & -32\alpha^2\sigma^2 \\
+16\sigma^4 & +64\alpha\beta\epsilon^2 & -64\alpha\beta\gamma^2 & -128\alpha\beta\mu\sigma & +16\beta^4 \\
-64\beta^2\epsilon\gamma & -32\beta^2\mu^2 & +32\beta^2\sigma^2 & +16\epsilon^4 & +32\epsilon^2\gamma^2 \\
-64\epsilon^2\mu\sigma & +64\epsilon\gamma\mu^2 & -64\epsilon\gamma\sigma^2 & +16\gamma^4 & +64\gamma^2\mu\sigma \\
+16\mu^4 & +32\mu^2\sigma^2 & & & \\
-8\gamma^2X^2 & -16\mu\sigma X^2 & +16\alpha\beta X^2 & +8\epsilon^2X^2 & \\
+X^4 & & & &
\end{array}\right)
$$

7.8 16-element group 8

An 8'th 16-element group is generated by the relations $2^2 = 0$, $1^2 = 2$, $8^2 = 1^{-1}$, $4^2 = 2$, $2 = 1^{-1}4^{-1}14$, and $1 = 8^{-1}4^{-1}84$. The product (Cayley) table resulting is

0	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*
1	*2*	*3*	*0*	*6*	*7*	*5*	*4*	*10*	*11*	*9*	*8*	*14*	*15*	*13*	*12*
2	*3*	*0*	*1*	*5*	*4*	*7*	*6*	*9*	*8*	*11*	*10*	*13*	*12*	*15*	*14*
3	*0*	*1*	*2*	*7*	*6*	*4*	*5*	*11*	*10*	*8*	*9*	*15*	*14*	*12*	*13*
4	*7*	*5*	*6*	*2*	*0*	*1*	*3*	*14*	*15*	*12*	*13*	*11*	*10*	*9*	*8*
5	*6*	*4*	*7*	*0*	*2*	*3*	*1*	*15*	*14*	*13*	*12*	*10*	*11*	*8*	*9*
6	*4*	*7*	*5*	*3*	*1*	*2*	*0*	*13*	*12*	*14*	*15*	*8*	*9*	*11*	*10*
7	*5*	*6*	*4*	*1*	*3*	*0*	*2*	*12*	*13*	*15*	*14*	*9*	*8*	*10*	*11*
8	*10*	*9*	*11*	*12*	*13*	*14*	*15*	*3*	*1*	*0*	*2*	*7*	*6*	*4*	*5*
9	*11*	*8*	*10*	*13*	*12*	*15*	*14*	*1*	*3*	*2*	*0*	*6*	*7*	*5*	*4*
10	*9*	*11*	*8*	*14*	*15*	*13*	*12*	*0*	*2*	*1*	*3*	*4*	*5*	*6*	*7*
11	*8*	*10*	*9*	*15*	*14*	*12*	*13*	*2*	*0*	*3*	*1*	*5*	*4*	*7*	*6*
12	*15*	*13*	*14*	*9*	*8*	*10*	*11*	*4*	*5*	*7*	*6*	*2*	*0*	*1*	*3*
13	*14*	*12*	*15*	*8*	*9*	*11*	*10*	*5*	*4*	*6*	*7*	*0*	*2*	*3*	*1*
14	*12*	*15*	*13*	*11*	*10*	*9*	*8*	*6*	*7*	*4*	*5*	*3*	*1*	*2*	*0*
15	*13*	*14*	*12*	*10*	*11*	*8*	*9*	*7*	*6*	*5*	*4*	*1*	*3*	*0*	*2*

This has a solution with 9 parameters.

0	0	0	0	0	0	0	0	...
0	0	0	0	$-\rho$	ρ	$-\lambda$	λ	...
0	0	0	0	0	0	0	0	...
0	0	0	0	ρ	$-\rho$	λ	$-\lambda$	...
0	ρ	0	$-\rho$	0	0	$-\alpha$	α	...
0	$-\rho$	0	ρ	0	0	α	$-\alpha$	...
0	λ	0	$-\lambda$	α	$-\alpha$	0	0	...
0	$-\lambda$	0	λ	$-\alpha$	α	0	0	...
0	0	0	0	ν	$\nu-\mu-\sigma$	$\sigma-\nu$	$\mu-\nu$	...
0	0	0	0	$\nu-\mu-\sigma$	ν	$\mu-\nu$	$\sigma-\nu$	...
0	0	0	0	$-\nu$	$\mu+\sigma-\nu$	$\nu-\sigma$	$\nu-\mu$	...
0	0	0	0	$\mu+\sigma-\nu$	$-\nu$	$\nu-\mu$	$\nu-\sigma$	...
0	μ	0	$-\mu$	β	τ	$-\tau$	$-\beta$	...
0	$-\mu$	0	μ	τ	β	$-\beta$	$-\tau$	...
0	σ	0	$-\sigma$	$-\beta$	$-\tau$	β	τ	...
0	$-\sigma$	0	σ	$-\tau$	$-\beta$	τ	β	...

...	0	0	0	0	0	0	0	0
...	0	0	0	0	$-\mu$	μ	$-\sigma$	σ
...	0	0	0	0	0	0	0	0
...	0	0	0	0	μ	$-\mu$	σ	$-\sigma$
...	$\nu-\mu-\sigma$	ν	$\mu+\sigma-\nu$	$-\nu$	$-\beta$	$-\tau$	β	τ
...	ν	$\nu-\mu-\sigma$	$-\nu$	$\mu+\sigma-\nu$	$-\tau$	$-\beta$	τ	β
...	$\mu-\nu$	$\sigma-\nu$	$\nu-\mu$	$\nu-\sigma$	τ	β	$-\beta$	$-\tau$
...	$\sigma-\nu$	$\mu-\nu$	$\nu-\sigma$	$\nu-\mu$	β	τ	$-\tau$	$-\beta$
...	0	0	0	0	$\rho-\lambda-\epsilon$	$-\epsilon$	$\lambda+\epsilon$	$-\rho+\epsilon$
...	0	0	0	0	$-\epsilon$	$\rho-\lambda-\epsilon$	$\epsilon-\rho$	$\lambda+\epsilon$
...	0	0	0	0	$\lambda+\epsilon-\rho$	ϵ	$-\lambda-\epsilon$	$\rho-\epsilon$
...	0	0	0	0	ϵ	$\lambda+\epsilon-\rho$	$\rho-\epsilon$	$-\lambda-\epsilon$
...	$-\epsilon$	$\rho-\lambda-\epsilon$	ϵ	$\lambda+\epsilon-\rho$	0	0	$-\alpha$	α
...	$\rho-\lambda-\epsilon$	$-\epsilon$	$\lambda+\epsilon-\rho$	ϵ	0	0	α	$-\alpha$
...	$\epsilon-\rho$	$\lambda+\epsilon$	$\rho-\epsilon$	$-\lambda-\epsilon$	α	$-\alpha$	0	0
...	$\lambda+\epsilon$	$\epsilon-\rho$	$-\lambda-\epsilon$	$\rho-\epsilon$	$-\alpha$	α	0	0

$$\begin{array}{lll} \alpha \to C_1 & \beta \to C_2 & \epsilon \to C_8 \\ \rho \to C_9 & \lambda \to C_3 & \mu \to C_4 \\ \nu \to C_6 & \sigma \to C_5 & \tau \to C_7 \end{array}$$

If we define new generators based on the old ones, the commutation relations are greatly simplified:

$$\begin{array}{ccc} D_1 = 1/4\,(C_2 + C_7) & D_2 = 1/4C_6 & D_3 = 1/4C_8 \\ D_4 = 2C_3 - C_8 & D_5 = C_1 & D_6 = C_4 + C_5 + C_6 \\ D_7 = C_4 - C_5 & D_8 = C_2 - C_7 & D_9 = C_8 + 2C_9 \end{array}$$

$$\begin{array}{ccc} [D_1, D_2] = -D_3 & [D_2, D_3] = D_1 & [D_3, D_1] = D_2 \\ [D_4, D_5] = 2D_9 & [D_4, D_6] = 4D_8 & [D_4, D_8] = -4D_6 \\ [D_4, D_9] = -8D_5 & [D_5, D_6] = -2D_7 & [D_5, D_7] = 2D_6 \\ [D_5, D_9] = 2D_4 & [D_6, D_7] = -4D_5 & [D_6, D_8] = 2D_4 \\ [D_7, D_8] = 2D_9 & [D_7, D_9] = -4D_8 & [D_8, D_9] = 4D_7 \end{array}$$

The transformation partitions neatly into a transformation based on D_1, D_2, and D_3, and a transformation based on the other 6.

The group has four one-dimensional and three two-dimensional representations. If I can partition the transformation based on 6 parameters into 2 based on three it may be possible to link the N-dimensional representations with the transformations. This does not seem to be possible. Either I have an error in the generator commutations or I must abandon the hypothesis that the transformations may be partitioned into transformations of the independent representations.

The Casimir invariant is simple:

$$-1/32C_1^2 + 1/32C_2^2 + 1/32C_3^2 - 1/64C_4^2 - 1/8C_5^2 - 1/16C_6^2 - 1/32C_7^2 - 1/16C_8^2 - 1/32C_9^2$$

7.9 16-element group 9

The ninth non-abelian 16-element group is generated by the following relations: $1^2 = 2$, $2^2 = 4$, $4^2 = 0$, $8^2 = 0$, $4 = 2^{-1}8^{-1}28$, and $2 = 1^{-1}8^{-1}18$. The resulting product (Cayley) table is:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	2	3	4	5	6	7	0	9	10	11	12	13	14	15	8
2	3	4	5	6	7	0	1	10	11	12	13	14	15	8	9
3	4	5	6	7	0	1	2	11	12	13	14	15	8	9	10
4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	6	7	0	1	2	3	4	13	14	15	8	9	10	11	12
6	7	0	1	2	3	4	5	14	15	8	9	10	11	12	13
7	0	1	2	3	4	5	6	15	8	9	10	11	12	13	14
8	11	14	9	12	15	10	13	0	3	6	1	4	7	2	5
9	12	15	10	13	8	11	14	1	4	7	2	5	0	3	6
10	13	8	11	14	9	12	15	2	5	0	3	6	1	4	7
11	14	9	12	15	10	13	8	3	6	1	4	7	2	5	0
12	15	10	13	8	11	14	9	4	7	2	5	0	3	6	1
13	8	11	14	9	12	15	10	5	0	3	6	1	4	7	2
14	9	12	15	10	13	8	11	6	1	4	7	2	5	0	3
15	10	13	8	11	14	9	12	7	2	5	0	3	6	1	4

The differential transform array is given by

$$
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & \chi & -\mu-\delta & -\chi & 0 & -\sigma & \mu+\delta & \sigma & . \\
0 & \mu & \sigma-\epsilon & -\mu & 0 & \nu & \epsilon-\sigma & -\nu & \\
0 & \epsilon & -\delta-\nu & -\epsilon & 0 & \sigma-\epsilon-\chi & \nu+\delta & \chi+\epsilon-\sigma & . \\
0 & \delta & \epsilon+\chi & -\delta & 0 & -\mu-\nu-\delta & -\epsilon-\chi & \mu+\delta+\nu & \\
0 & -\sigma & \mu+\delta & \sigma & 0 & \chi & -\mu-\delta & -\chi & . \\
0 & \nu & \epsilon-\sigma & -\nu & 0 & \mu & \sigma-\epsilon & -\mu & \\
0 & \sigma-\epsilon-\chi & \nu+\delta & \chi+\epsilon-\sigma & 0 & \epsilon & -\nu-\delta & -\epsilon & . \\
0 & -\mu-\delta-\nu & -\epsilon-\chi & \mu+\delta+\nu & 0 & \delta & \epsilon+\chi & -\delta &
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
. & -\sigma & \mu & \sigma-\epsilon-\chi & \delta & \chi & \nu & \epsilon & -\mu-\delta-\nu \\
 & -\mu-\delta & \epsilon-\sigma & -\delta-\nu & -\chi-\epsilon & \mu+\delta & \sigma-\epsilon & \delta+\nu & \epsilon+\chi \\
. & \sigma & -\mu & -\sigma+\epsilon+\chi & -\delta & -\chi & -\nu & -\epsilon & \mu+\delta+\nu \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
. & \chi & \nu & \epsilon & -\mu-\delta-\nu & -\sigma & \mu & \sigma-\epsilon-\chi & \delta \\
 & \mu+\delta & \sigma-\epsilon & \delta+\nu & \epsilon+\chi & -\mu-\delta & \epsilon-\sigma & -\delta-\nu & -\chi-\epsilon \\
. & -\chi & -\nu & -\epsilon & \mu+\delta+\nu & \sigma & -\mu & -\sigma+\epsilon+\chi & -\delta \\
 & 0 & -\alpha & -\beta & \alpha & 0 & -\gamma & \beta & \gamma \\
. & \gamma & 0 & -\alpha & -\beta & \alpha & 0 & -\gamma & \beta \\
 & \beta & \gamma & 0 & -\alpha & -\beta & \alpha & 0 & -\gamma \\
. & -\gamma & \beta & \gamma & 0 & -\alpha & -\beta & \alpha & 0 \\
 & 0 & -\gamma & \beta & \gamma & 0 & -\alpha & -\beta & \alpha \\
. & \alpha & 0 & -\gamma & \beta & \gamma & 0 & -\alpha & -\beta \\
 & -\beta & \alpha & 0 & -\gamma & \beta & \gamma & 0 & -\alpha \\
 & -\alpha & -\beta & \alpha & 0 & -\gamma & \beta & \gamma & 0
\end{array}
$$

The commutation relations among the generators are given in the following table. $\delta V = \alpha A + \beta B + \gamma C + \delta D + \mu M + \nu N + \sigma S + \epsilon E + \chi Q$.

$[S, M] = -2A$	$[S, E] = -2B$	$[S, Q] = -2B$	$[S, D] = -2A + 2G$
$[S, N] = -2A$	$[S, A] = -M + 2D - N$	$[S, B] = S + E + Q$	$[S, G] = -M - N$
$[M, E] = -2A + 2G$	$[M, Q] = -2A$	$[M, D] = 2B$	$[M, N] = 2B$
$[M, A] = S - Q$	$[M, B] = -M + D + N$	$[M, G] = S + 2E - Q$	$[E, Q] = 2B$
$[E, D] = 2A - 2G$	$[E, N] = 0$	$[E, A] = -M - D + N$	$[E, B] = -2S - 2Q$
$[E, G] = M + D - N$	$[Q, D] = 0$	$[Q, N] = 2G$	$[Q, A] = -D + 2N$
$[Q, B] = -S + E - Q$	$[Q, G] = 2M - D$	$[D, N] = 2B$	$[D, A] = S - E + Q$
$[D, B] = -2M + 2N$	$[D, G] = -S + E - Q$	$[N, A] = 2S + E$	$[N, B] = -M - D + N$
$[N, G] = E - 2Q$	$[A, B] = 0$	$[A, G] = 0$	$[B, G] = 0$