Transformations over a space defined over a finite group

James N. Bellinger

March 22, 2017

Abstract

Continuous transformations of dihedral groups of >= 6 have a maximum of $3(p-1)/2$ independent parameters if p is odd, and $3(p-2)/2$ if p is even.

The transformations T_H of a group H which is a subgroup of G will sometimes be a subset of the transformations T_G over G, but not in general. In particular, if $G = H \otimes Z_2$, T_H is a subset of T_G. Therefore in general, if a subgroup H of G transforms as T_{GH} as part of T_G, T_{GH} will not be the same as the transforms T_H of H as a stand-alone group.

Unless the original group objects are physical objects, these continuous transformations are unlikely to have physical meaning.

1 Introduction

As described in an earlier paper [1], non-abelian finite groups can sustain continuous transformations in a space that mixes the group elements, which maintain the original group structure. These transformations are of the form of $n \times n$ matrices, where n is the order of the group. Name such a transform matrix V; the equation governing its differential changes is

$$\delta V_{i,kj^{-1}} + \delta V_{j,i^{-1}k} = \delta V_{ij,k}$$

. The i, j, and k are the numbered group elements, such that (e.g.) kj^{-1} is the number of the element given by the group product of group element k and the inverse of j.

While it is possible to solve for the generators of such a transformation for any given group, it is good to have general solutions available. One simple example is dihedral groups.

2 Dihedral groups: $2\times$ p

Dihedral groups of order 2p obey $a^2 = 0$ and $b^p = 0$ and $ba = ab^{-1}$ for appropriate selections of a and b. It is convenient to number the group elements from $\{0 \ldots 2p-1\}$ mapping $i \to b^i$

[1]Symmetries of preon interactions modeled as a finite group. J.N. Bellinger J.Math.Phys.38:3414-3426,1997

for $i \in \{0 \ldots p-1\}$ and $i \to ab^{i-p}$ for $i \in \{p \ldots 2p-1\}$. Name the first subset A and the second one B. The subset A is an abelian subgroup, and we already know that the $\delta V_{i,j}$ components are all 0 when i and j are in the abelian subgroup.

The δV array then neatly partitions into 4 sub-matrices:

$$\delta V = \begin{pmatrix} V1 & V2 \\ V3 & V4 \end{pmatrix} \tag{1}$$

$$\delta V_{i,kj^{-1}} + \delta V_{j,i^{-1}k} = \delta V_{ij,k}$$

Case	i	j	k
1	A	A	A
2	A	A	B
3	A	B	A
4	A	B	B
5	B	A	B
6	B	A	B
7	B	B	A
8	B	B	B

For Case 1, all the $\delta V_{x,y}$ have x and y in subset A, which is an abelian subgroup and for which all the δV $V1$ entries are zero.

Case	$\delta V_{i,kj^{-1}}$	$\delta V_{j,i^{-1}k}$	$\delta V_{ij,k}$
2	$V2$	$V2$	$V2$
3	$V2$	$V3$	$V3$
4	$V1 = 0$	$V4$	$V4$
5	$V3$	$V2$	$V3$
6	$V4$	$V1 = 0$	$V4$
7	$V4$	$V4$	$V1 = 0$
8	$V3$	$V3$	$V2$

$V2$ and $V3$ are, of course intimately linked, and $V4$ is not related to either of them.

Using the notation described above, if i and j are in A their combination ij is $mod(i+j)_p$. For simplicity I will denote $mod(i+j)_p$ as $\{i+j\}$. If $i \in A$, i^{-1} we can represent as $N-i$ or simply $\{-i\}$ for convenience. If $i \in B$, $i^{-1} = i$ as each element is its own inverse. When $i, j \in B$ (i.e. i is really $mod(i)_N + N \equiv \{i\} + p$), their combination ij is $\{j-i\} + p$.

$i \in$	$j \in$	ij
A	A	$\{i+j\} \in A$
A	B	$\{j-i\} + p \in B$
B	A	$\{i-j\} + p \in B$
B	B	$\{j-i\} \in A$

2.1 V4

2.1.1 Case 7

In this case $i \in B$, $j \in B$, and $k \in A$. Since $ij \in A$, the right-hand side of this equation is zero.

$$\delta V_{i,kj^{-1}} + \delta V_{j,i^{-1}k} = \delta V_{ij,k} = 0$$

$$\delta V_{i,kj^{-1}} = \delta V_{i,\{j-k\}+p} = \delta V_{\{i\}+p,\{j-k\}+p}$$

$$\delta V_{j,ik} = \delta V_{j,\{i+k\}+p} = \delta V_{\{j\}+p,\{i+k\}+p}$$

The subscripts are hard to read, so define

$$\delta V_{x,y} \equiv ((x,y))$$

and we have

$$((\{i\}+p,\{j-k\}+p)) = -((\{j\}+p,\{i+k\}+p)) \tag{2}$$

We already know we're in the $V4$ block, so we can omit the $+p$ for clarity here.

$$((\{i\},\{j-k\})) = -((\{j\},\{i+k\}))$$

We can use this more easily if we define $\{J\} \equiv \{j-k\}$, which gives use

$$((\{i\},\{J\})) = -((\{J+k\},\{i+k\}))\forall k \tag{3}$$

Wrap-around diagonals have the same value. For example, if $\{i\} = 2$ and $\{J\} = 1$ has $((\{1\},\{1\})) = -x$, then in $V4$ we have:

$$\begin{array}{ccccccc}
0 & x & 0 & 0 & \dots & 0 & -x \\
-x & 0 & x & 0 & \dots & 0 & 0 \\
0 & -x & 0 & x & \dots & 0 & 0 \\
0 & 0 & -x & 0 & \dots & 0 & 0 \\
 & & & & \dots & & \\
0 & 0 & 0 & 0 & \dots & 0 & x \\
x & 0 & 0 & 0 & \dots & -x & 0
\end{array} \tag{4}$$

There are p partial diagonals, which are paired. The central diagonal is zero. If p is odd there can be $(p-1)/2$ of these ($b^p = 0$) set of diagonals. If p is even, then we can use Equation 3 and set $i = 0$, $J = p/2$, and use $k = p/2$ to find $((0,p/2)) = -((0,p/2))$. Clearly all elements in this partial diagonal are also 0. So if p is even the number of paired diagonals is $(p-2)/2$ and if p is odd there are $(p-1)/2$.

2.1.2 Case 6

In this case $i \in B$, $j \in A$, and $k \in B$. Since j and $i^{-1}k$ are both in A, the second term is 0 and the fundamental equation is now:

$$\delta V_{i,kj^{-1}} + 0 = \delta V_{ij,k}$$

Once again this involves elements of $V4$

$$((\{i\} + p, \{k - j\} + p)) = ((\{i + j\} + p, \{k\} + p))$$

If we drop the p for clarity, and define $\{K\} = \{k - j\}$ then we have

$$((\{i\}, \{K\})) = ((\{i + j\}, \{K + j\}))\forall j$$

This simply tells us that all elements in a wrap-around diagonal are the same in $V4$, which we knew already from Case 7.

2.1.3 Case 4

In this case $i \in A$, $j \in B$, and $k \in B$. Since both i and kj^{-1} are in A, the first term is 0 and the fundamental equation is now:

$$0 + \delta V_{j,i^{-1}k} = \delta V_{ij,k}$$

This also involves elements of $V4$:

$$((\{j\} + p, \{i + k\} + p)) = ((\{j - i\} + p, \{k\} + p))$$

If we drop the p for clarity, and define $\{J\} = \{j - i\}$, then we have

$$((\{J + i\}, \{k + i\})) = ((\{J\}, \{k\}))$$

This simple tells us that all elements in a wrap-around diagonal are the same in $V4$, which we knew already from Case 7 and Case 6. Simple parameter substitution shows these 3 sets of equations are equivalent.

2.1.4 Calculations within V4

Since Cases 4, 6, and 7 exhaust the equations that involve elements of $V4$, there are no further restrictions on the number of independent solutions and we have $(p-1)/2$ (or $(p-2)/2$ if p is even) generators which can be represented by the following block array, where 0 is an $p \times p$ array of 0 and $V4$ is a wrap-around diagonal array like that in Eq 4. For purposes of categorization we can without loss of generality set x=1. The wrap-arounds start in positions $((\{1\}, \{2\}))$ (for +1) and $((\{2\}, \{1\}))$ (for -1) for the first array, $((\{1\}, \{3\}))$ (for +1) and

$((\{3\},\{1\}))$ (for -1) for the second array, and so on up to $((\{1\},\{p+1)/2\}))$ (for +1) for the $(p-1)/2)$'th array. Give the one with $((\{1\},\{m+1\}))=1$ the name G_m.

This is easier to deal with if we define an array

$$P = \begin{cases} P_{i,j} = 1 \text{ for } j = i+1 \\ P_{i,j} = 1 \text{ for } i = p, j = 1 \\ P_{i,j} = 0 \text{ otherwise} \end{cases} \tag{5}$$

We can define our $(p-1)/2)$ generator representations as

$$G_m = P^m - P^{p-m} \text{ for } m \in \{1 \ldots (p-1)/2\} \tag{6}$$

Obviously $G_0 = 0$, and if p is even $G_{p/2} = 0$ too. From this construction it is easy to see that these generator representations always commute.

$$G_m G_n = G_n G_m \tag{7}$$

It further follows (recalling that $P^p = I$), and considering $3m$ as being modulo p, that

$$\begin{aligned} {G_m}^3 &= P^{3m} - 3P^{2m}P^{p-m} + 3P^m P^{2p-2m} - P^{3p-3m} && (8) \\ &= (P^{3m} - P^{p-3m}) - 3(P^m - P^{p-m}) && (9) \\ &= G_{3m} - 3G_m && (10) \end{aligned}$$

If p is odd, or if p is even and $m \neq p/2$, $3m$ will always refer to a different generator representation than m. Note that if $p > 3m > (p-1)/2$ then $G_{3m} = -G_{p-3m}$

$$G_m G_n G_l = G_{m+n+l} + G_{m-n-l} + G_{l-m-n} + G_{n-m-l} \tag{11}$$

When $p = 3$ or $p = 4$, there is only one generator in the representation of $V4$. When $p = 5$ there are two, and it is straightforward to design linear combinations L_1 and L_2 such that ${L_1}^3 = \alpha L_1$ and ${L_2}^3 = \beta L_2$. For $L_1 = c_1 G_1 + c_2 G_2$, the ratio c_1/c_2 is one of $(1 \pm \sqrt(5))/2$ or $(-2 \pm \sqrt(5))$.

For calculations define

$$H_m = P^m + P^{p-m} \tag{12}$$

$H_0 = 2I$, $H_m = H_{-m}$. Even products of G's involve only the H's, and odd products only involve the G's.

2.2 V2, V3

From $\delta V_{i,j} = -\delta V_{j^{-1},i^{-1}}$ we know that these two sub-matrices are not independent. Cases 2, 3, and 5, when re-written to reflect only elements in $V2$, are equivalent sets of equations. For simplicity of notation if $k' > p$ I use the $k \equiv k' - p$, element $k' = ab^k$ below.

$$\delta V_{b^i,ab^{k-j}} + \delta V_{b_j,ab^{k+i}} = \delta V_{b^{i+j},ab^k}$$

$$\delta V_{b^i,ab^{j-k}} - \delta V_{b^{i-k},ab^j} = -\delta V_{b^{-k},ab^{j-i}}$$

$$-\delta V_{b^{j-k},ab^i} + \delta V_{b^j,ab^{k+i}} = -\delta V_{b^{-k},ab^{i+j}}$$

Keeping only the exponents of b and taking the presence of a in the second place as given, I write the first equation (equivalent to the other two) as

$$\langle i, k-j\rangle + \langle j, k+i\rangle = \langle i+j, k\rangle \tag{13}$$

Equation Case 8 is not equivalent to the other three (2, 3, and 5). When I put it in a form that references only $V2$ I get

$$\langle k-j, i\rangle + \langle i-k, j\rangle + \langle j-i, k\rangle = 0 \tag{14}$$

Here i, j, and k range from 0 to $p-1$

If you set $j = i$ in Equation 14 you can easily see that

$$\langle x, y\rangle = -\langle -x, y\rangle \tag{15}$$

If p is odd the above are all distinct except where $x = 0$, and we know already that $\langle 0, y\rangle = 0$. If p is even, then $x = p/2 = -x$ and $\langle p/2, y\rangle = -\langle p/2, y\rangle$ and these are also clearly 0 for all y: a second row of 0.

It isn't hard to show that Equation 14 is a consequence of Equation 15 and Equation 13, so we use the latter two as a simpler pair.

It is convenient to use Equation 13 in a slightly different form, defining $H = i + j$ to get

$$\langle H, k\rangle = \langle i, k-H+i\rangle + \langle H-i, k+1\rangle \tag{16}$$

Let $H = p-1$. From Equation 15 if we know $\langle p-1, k\rangle$ we know $\langle 1, k\rangle$. Since the sums of each row or column of the differential matrix is 0, we also have

$$\langle p-1, 0\rangle = \sum_{j=1}^{p-1} \langle p-1, j\rangle \tag{17}$$

and using Equation 15 we find

$$\langle p-1, k\rangle = \langle i, k+i+1\rangle - \langle i+1, k+1\rangle \tag{18}$$

Let $I = 1$.

$$\langle p-1, k\rangle = \langle 1, k+2\rangle - \langle 2, k+1\rangle = -\langle p-1, k+2\rangle - \langle 2, k+1\rangle \tag{19}$$

We can rewrite this as

$$\langle 2, k+1\rangle = -\langle p-1, k\rangle + \langle p-1, k+2\rangle \tag{20}$$

Since this is true for all k, we now have each element of the row $\langle 2, x\rangle$ In terms of the $p-1$ remaining elements of the row $\langle p-1, x\rangle$. This automatically gives us the row $\langle p-2, x\rangle$.

Let $I = 2$. The same sort of manipulation shows us

$$\langle p-1, k\rangle = \langle 2, k+3\rangle - \langle 3, k+2\rangle \tag{21}$$

Since we know the $\langle 2, x\rangle$ in terms of the top row, we can solve for $\langle 3, x\rangle$ also, and plainly iterate through each of the rows in turn.

All rows are expressible in terms of the elements of the top row, and one of those elements in the top row in turn is expressible in terms of the rest of the row, so there are a maximum of $p-1$ independent variables if p is odd, and therefore a maximum of $p-1$ generators involving the off-diagonal arrays.

Explicit calculations on examples suggest that $p-1$ is also the minimum, which, combined with the generators from the diagonal matrix, says there are $3(p-1)/2$ generators for the transformations of a finite group of order $2p$.

We can produce neater equations. Let $H = p-2$ and $i = p-1$.

$$\langle p-2, k\rangle = \langle p-1, k+1\rangle + \langle p-1, k-1\rangle \tag{22}$$

Then, successively, let $H = p-3$ and $i = p-2$, then $H = p-4$ and $i = p-3$ to get

$$\langle p-3, k\rangle = \langle p-1, k+2\rangle + \langle p-1, k\rangle + \langle p-1, k-2\rangle \tag{23}$$

$$\langle p-4, k\rangle = \langle p-1, k+3\rangle + \langle p-1, k+1\rangle + \langle p-1, k-1\rangle + \langle p-1, k-3\rangle \tag{24}$$

and so on.

$$\langle p-X, k\rangle = \sum_{a=1-X}^{X-1,(2)} \langle p-1, k+a\rangle \tag{25}$$

This makes it convenient to deal with the case where p is even. Recall that

$$\langle p/2, k\rangle = 0$$

. In Equation 25 set $X = p/2$. This gives us

$$\langle p-p/2, k\rangle = \langle p/2, k\rangle = 0 = \sum_{a=1-p/2}^{p/2-1,(2)} \langle p-1, k+a\rangle \tag{26}$$

The right hand side consists of $(p-2)/2+1 = p/2$ terms, or half of the $\langle p-1, k\rangle$ row. If k is odd these are all the odd numbered elements, and if k is even all the even numbered elements of the row. This gives 2 equations: all even numbered elements of the $p-1$ row sum to zero, and so do all the odd numbered elements. Adding both says that the sum of all elements in the row is zero, which we already knew. Therefore there are 2 independent constraints on the values of the elements of the $p-1$ row (from which all other rows can be determined), so there are $p-2$ independent parameters in the transformation over the off-diagonal sub-arrays.

Combining the diagonal sub-matrix results and the off-diagonal sub-matrix results tells us that for a dihedral group of order p, there are a maximum of $3(p-1)/2$ independent parameters in the transformation if p is odd, and $3(p-2)/2$ if p is even.

In all cases I have solved the maximum number of parameters is the total number of parameters.

3 H a subgroup of G

We use the differential continuous transformation equation

$$\delta V_{i,kj^{-1}} + \delta V_{j,i^{-1}k} = \delta V_{ij,k} \tag{27}$$

For simplicity of notation, take the δV as given and represent the above as

$$(i, kj^{-1}) + (j, i^{-1}k) = (ij, k) \tag{28}$$

Let T_H be the continuous transformations (defined in [2]) over H. Let H be a subgroup of G.

Is $T_H \leq T_G$?

3.1 H is a normal subgroup of G

3.1.1 $G = H \otimes Z_2$

For this case, $T_H \leq T_G$, as I show below.

The cosets of G are H and Q, where $Q = Hb$ and $b^2 = 0$, with 0 representing the identity. Retain the ordering, so that each element h_b of Q is hb where $h \in H$. Since b commutes with everything I will place it at the right.

The i, j, and k will be in either H or Q, giving us 8 sets of equations. The transformation array partitions into 4 blocks. A and D are on the diagonal, B and C off-diagonal.

$$\begin{array}{ccc} & H & Q \\ H & A & B \\ Q & C & D \end{array}$$

[2] Journal of Mathematical Physics 38:3414-3426, 1997

The list of equations is

	i	j	k	*Generic*	*Block*	*Detail*
1	H	H	H	$(H,H)+(H,H)=(H,H)$	$A+A\in A$	$(i,kj^{-1})+(j,i^{-1}k)=(ij,k)$
2	H	H	Q	$(H,Q)+(H,Q)=(H,Q)$	$B+B\in B$	$(i,kj^{-1}b)+(j,i^{-1}kb)=(ij,kb)$
3	H	Q	H	$(H,Q)+(Q,H)=(Q,H)$	$B+C\in C$	$(i,kj^{-1}b)+(jb,i^{-1}k)=(ijb,k)$
4	H	Q	Q	$(H,H)+(Q,Q)=(Q,Q)$	$A+D\in D$	$(i,kj^{-1})+(jb,i^{-1}kb)=(ijb,kb)$
5	Q	H	H	$(Q,H)+(H,Q)=(Q,H)$	$C+B\in C$	$(ib,kj^{-1})+(j,i^{-1}kb)=(ijb,k)$
6	Q	H	Q	$(Q,Q)+(H,H)=(Q,Q)$	$D+A\in D$	$(ib,kj^{-1}b)+(j,i^{-1}k)=(ijb,kb)$
7	Q	Q	H	$(Q,Q)+(Q,Q)=(H,H)$	$D+D\in A$	$(ib,kj^{-1}b)+(jb,i^{-1}kb)=(ij,k)$
8	Q	Q	Q	$(Q,H)+(Q,H)=(H,Q)$	$C+C\in B$	$(ib,kj^{-1})+(jb,i^{-1}k)=(ij,kb)$

Notice that equation sets 1, 4, 6, and 7 include only A and D, and the rest only include B and C. Equation set 1, if in isolation, would reproduce T_H. Since 2, 3, 5, and 8 do not involve these elements, it suffices to show that equations 1, 4, 6, and 7 will in fact continue to reproduce T_H.

Add up equation sets 4, 6, and 7. This gives

$$(i,kj^{-1})+(j,i^{-1}k)+2(jb,i^{-1}kb)+2(ib,kj^{-1}b)=(ij,k)+2(ijb,kb) \tag{29}$$

which of course simplifies to

$$(jb,i^{-1}kb)+(ib,kj^{-1}b)=(ijb,kb) \tag{30}$$

Add equation sets 4 and 6. This gives

$$(i,kj^{-1})+(jb,i^{-1}kb)+(ib,kj^{-1}b)+(j,i^{-1}k)=2(ijb,kb) \tag{31}$$

which we combine with the above equation to find that

$$(i,kj^{-1})+(j,i^{-1}k)=(ijb,kb) \tag{32}$$

Solving for the rest of them is trivial, and we see that

$$(ijb,kb)=(ij,k) \quad (ib,kj^{-1}b)=(i,kj^{-1}) \quad (jb,i^{-1}kb)=(j,i^{-1}k) \tag{33}$$

In this case we see that the transformations within D exactly mirror those within A, which means that the same T_H, extended to reach the elements in Q, is part of the set of transformations over $H\otimes Z_2$.

For the sake of completeness, recalling that $(x,y)=-(y^{-1},x^{-1})$, we can show that the off-diagonal blocks are governed by

$$-(jk^{-1}b,i^{-1})=(i,kj^{-1}b)=(ib,kj^{-1})=-(jk^{-1},i^{-1}b) \tag{34}$$
$$-(k^{-1}b,j^{-1}i^{-1})=(ij,kb)=(ijb,k)=-(k^{-1},j^{-1}i^{-1}b) \tag{35}$$
$$-(k^{-1}ib,j^{-1})=(j,i^{-1}kb)=(jb,i^{-1}k)=-(k^{-1}i,j^{-1}b) \tag{36}$$

3.1.2 $H \otimes g$

Order the elements of g in some arbitrary order that has the identity as 0. Call the q'th element g_q.

The group G partitions into cosets which we can label as $C_q = Hg_q$.

Recall the fundamental equation:

$$(i, kj^{-1}) + (j, i^{-1}k) = (ij, k) \tag{37}$$

If the $\{i, j, k\} \in H$, we have the transformation T_H over H, provided it is not constrained by any other terms.

In the following cases, the $\{i, j, k\} \in H$ is already accounted for, and $q \neq 0$.

If the first term involves elements of (H, H), then we have $i \in H$, and if $k \in C_q$, then we must have $j \in C_q$. This means that the second and third terms must be in (C_q, C_q) and (C_q, C_q) respectively. $(H, H) + (C_q, C_q) \in (C_q, C_q)$.

If the second term involves elements of (H, H), then we have $j \in H$, and if $k \in C_q$, then we must have $i \in C_q$. This means that the first and third terms must be in (C_q, C_q) and (C_q, C_q) respectively. $(C_q, C_q) + (H, H) \in (C_q, C_q)$.

If the third term involves elements of (H, H), then we have $k \in H$, and if $j \in C_q$, then we must have $i \in C_q^{-1}$. This means that the first and second terms must be in (C_q^{-1}, C_q^{-1}) and (C_q^{-1}, C_q^{-1}) respectively. $(C_q^{-1}, C_q^{-1}) + (C_q^{-1}, C_q^{-1}) \in (H, H)$. Recalling that $(i, j) = -(j^{-1}, i^{-1})$, we can have all these equations involving (C_q, C_q), which simplifies the argument below.

We have $1 + 3 \times (n - 1)$ equation sets: One that mixes only (H, H) terms, and $(n - 1)$ sets of 3 equation sets mixing only (H, H) and (C_q, C_q) terms for a given q.

However, though each of the of the $(n-1)$ sets of equation sets is identical in form to each other, they are not as simple as the Z_2 example above. Since $q^2 \neq 0$ in general, though the first two equation sets are the same, the last, in order to involve only C_q elements, reverses the order.

$$(i, kj^{-1}) + (jq, i^{-1}kq) = (ijq, kq) \tag{38}$$

$$(iq, kj^{-1}q) + (j, i^{-1}k) = (ijq, kq) \tag{39}$$

$$\begin{aligned} (iq, kj^{-1}q) + (jq^{-1}, i^{-1}kq^{-1}) &= (ij, k) \\ (iq, kj^{-1}q) - (k^{-1}iq, j^{-1}q) &= (ij, k) \end{aligned} \tag{40}$$

Combining all three reduces to the following, but without more information we cannot solve for the (C_q, C_q) terms in terms of the (H, H) ones, and in general the interference between terms will mean that the T_H will no longer be a subset of T_G.

$$2(iq, kj^{-1}q) + (jq, i^{-1}kq) - (k^{-1}iq, j^{-1}q) = 2(ijq, kq) \tag{41}$$

3.1.3 Normal subgroup

The mixing is even more thorough if H is a generic normal subgroup of G. The C_q we can still define similarly, with $\{q\}$ being one element of each coset, but since in general $iq \neq qi$, the sets of equation sets are more complicated:

$$(i, kj^{-1}) + (jq, i^{-1}kq) = (ijq, kq) \tag{42}$$
$$(iq, kqj^{-1}) + (j, i^{-1}k) = (iqj, kq) \tag{43}$$
$$\begin{aligned}(iq, kqj^{-1}) + (jq^{-1}, q^{-1}i^{-1}k) &= (iqjq^{-1}, k) \\ (iq, kqj^{-1}) - (k^{-1}iq, qj^{-1}) &= (iqjq^{-1}, k)\end{aligned} \tag{44}$$

Clearly the equation sets for simple groups will be even further entangled, and T_H will not be a subset of T_G except by accident.

3.1.4 Consequence

Taking this in reverse, we see that the transformations of H when it is a subset of G will not in general be the same as the transformations of H taken standalone, because of the extra constraints due to its embedding in G.

This means that you cannot simply solve a larger group and automatically retrieve the transformations of its subgroups as they would be in isolation: each group needs to be studied on its own.

4 Physics applications

Can the symmetries of this type have physics applications?

If the group in question is a representation of physical entities whose interactions are faithfully represented by the group's Cayley table, then yes, it could. The original JMP paper studied the possibility that preon interactions could be modeled this way.

If, on the other hand, the group is the set of operators on some physical system, then the answer is no. This is because the set of operators which preserves the symmetries of that system will not include the *additions* of two operators. This creates a new operator which is not in the original set of symmetry-preserving operators. For example, in the usual representation, group element interactions are matrix multiplications, but when you treat them as basis vectors for a field, that introduces matrix addition, which creates matrices which don't have the same properties as the original ones.

You are invited to verify this yourself. An equilateral triangle can be operated on by 3 flips $\{F_1, F_2, F_3\}$ and 3 rotations $\{R_0, R_{120}, R_{240}\}$. A set of 3 2-points representing the triangle is operated on by $2x2$ matrices representing the operators. One infinitesimal transformation over F_1 turns out to be $F_1 + \epsilon F_2 - \epsilon F_3$. You quickly see that the result doesn't preserve any of the points, or transform them into each other.

In addition, there are two distinct entities which can represent null. The identity element, modulo some scaling, is one candidate, but so is the true zero, in which all coefficients of the group elements are zero. Notice that for two elements a and b such that $a^2 = b^2$ and $ab = ba$, the expanded product of $a + b$ and $a - b$ is true zero, even though neither of the terms is.

5 Examples

As described in the earlier paper, the continuous transformations over the dihedral group of order 6 (the smallest non-abelian finite group) are isometric to SU(2). The transformations over the alternating group A_4 (a 12-element group) are isometric to S(3).