Non-Linear Theory of Particle Acceleration at Astrophysical Shocks

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First Order Fermi Acceleration: a Primer

The particle is always advected back to the shock. The particle may either diffuse back to the shock or be advected downstream.

\[ \int d\mu \ P_u(\mu_0, \mu) d\mu = 1 \]  
Return Probability from UP=1

\[ \int d\mu \ P_d(\mu_0, \mu) d\mu < 1 \]  
Return Probability from DOWN<1

\[ N(E) = N_0 \left( \frac{p}{p_0} \right)^{-\gamma - 1} \]
\[ \gamma = -\frac{\log P}{\log G} \approx \frac{3}{r - 1} \]

P  Total Return Probability from DOWN
G  Fractional Energy Gain per cycle
r  Compression factor at the shock
The Return Probability and Energy Gain for Non-Relativistic Shocks

At zero order the distribution of (relativistic) particles downstream is isotropic: \( f(\mu) = f_0 \)

Return Probability = Escaping Flux/Entering Flux

\[
\Phi_{\text{out}} = -\int f_0(u_2 + \mu) \, d\mu = \frac{1}{2} (1 - u_2)^2
\]

\[
\Phi_{\text{i}} = \int f_0(u_2 + \mu_0) \, d\mu_0 = \frac{1}{2} (1 + u_2)^2
\]

\[
P_{\text{return}} = \frac{(1 - u_2)^2}{(1 + u_2)^2} \approx 1 - 4u_2
\]

The extrapolation of this equation to the relativistic case would give a return probability tending to zero! The problem is that in the relativistic case the assumption of isotropy of the function \( f \) loses its validity.

\[
G = \frac{\Delta E}{E} = \frac{4}{3} (u_1 - u_2)
\]

SPECTRAL SLOPE

\[
\gamma = -\frac{\log P_{\text{return}}}{\log G} = \frac{\log (1 - 4u_2)}{\frac{4}{3} (u_1 - u_2)} \approx \frac{3}{r - 1}
\]
The Diffusion-Convection Equation: A more formal approach

\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial x} \left[ D \frac{\partial f}{\partial x} \right] - u \frac{\partial f}{\partial x} + \frac{1}{3} \frac{du}{dx} \frac{\partial f}{\partial p} + Q(x, p, t)
\]

The solution is a power law in momentum

\[
f_0(p) = \frac{3u_1}{u_1 - u_2} \frac{N_{\text{inj}}}{4\pi p_{\text{inj}}^2} \left( \frac{p}{p_{\text{inj}}} \right)^{-3u_1 \frac{u_1}{u_1 - u_2}}
\]

The slope depends ONLY on the compression ratio (not on the diffusion coeff.)

Injection momentum and efficiency are free param.
The Need for a Non-Linear Theory

- The relatively large efficiency may break the Test Particle Approximation... What happens then? Cosmic Ray Modified Shock Waves

- Non-linear effects must be invoked to enhance the acceleration efficiency (problem with $E_{\text{max}}$)

Self-Generation of Magnetic Field and Magnetic Scattering
Going Non Linear: Part I

Particle Acceleration in the
Non Linear Regime: Shock Modification
Why Did We think About This?

- Divergent Energy Spectrum

\[ E_{CR} = \int dE E N(E) \propto \ln \left( \frac{E_{max}}{E_{min}} \right) \]

At Fixed energy crossing the shock front \( \rho u^2 \) tand at fixed efficiency of acceleration there are values of Pmax for which the integral exceeds \( \rho u^2 \) (absurd!)
If the few highest energy particles that escape from upstream carry enough energy, the shock becomes dissipative, therefore more compressive.

If Enough Energy is channelled to CRs then the adiabatic index changes from 5/3 to 4/3. Again this enhances the Shock Compressibility and thereby the Modification.
The Basic Physics of Modified Shocks

\[ \rho(x)u(x) = \rho_0 u_0 \quad \text{Conservation of Mass} \]

\[ \rho_0 u_0^2 + P_{g,0} = \rho(x)u(x)^2 + P_g(x) + P_{CR}(x) \quad \text{Conservation of Momentum} \]

\[ \frac{\partial f}{\partial t} = \frac{\partial}{\partial x} \left[ D \frac{\partial f}{\partial x} \right] - u \frac{\partial f}{\partial x} + \frac{1}{3} \frac{du}{dx} p \frac{\partial f}{\partial p} + Q(x, p, t) \quad \text{Equation of Diffusion} \]

Convection for the Accelerated Particles
Main Predictions of Particle Acceleration at Cosmic Ray Modified Shocks

- Formation of a Precursor in the Upstream plasma
- The Total Compression Factor may well exceed 4. The Compression factor at the subshock is <4
- Energy Conservation implies that the Shock is less efficient in heating the gas downstream
- The Precursor, together with Diffusion Coefficient increasing with $p \rightarrow$ NON POWER LAW SPECTRA!!! Softer at low energy and harder at high energy
Spectra at Modified Shocks

Very Flat Spectra at high energy

Amato and PB (2005)
Efficiency of Acceleration (PB, Gabici & Vannoni (2005))

Note that only this Flux ends up DOWNSTREAM!!!

This escapes out To UPSTREAM

\[ F'_{\text{tot}} \]
\[ F'_{E} \]
\[ F'_{\text{adv}} \]

\[ \frac{F}{1/2\rho_0 u^3_0} \]

\[ u_0 = 5 \times 10^8 \text{ cm/s} \]
\[ p_{\text{max}}/mc = 10^6 \]
\[ \xi = 3.5 \]
Suppression of Gas Heating

PB, Gabici & Vannoni (2005)

The suppressed heating might have already been detected (Hughes, Rakowski & Decourchelle (2000))
Going Non Linear: Part II

Coping with the Self-Generation of Magnetic field by the Accelerated Particles

Basic Assumptions:
1. The Spectrum is a power law
2. The pressure contributed by CR's is relatively small
3. All Accelerated particles are protons

The basic physics is in the so-called streaming instability:

of particles that propagates in a plasma is forced to move at speed smaller or equal to the Alfven speed, due to the excitation of Alfven waves in the medium.
Pitch angle scattering and Spatial Diffusion

The Alfven waves can be imagined as small perturbations on top of a background B-field

\[ \vec{B} = \vec{B}_0 + \vec{B}_1 \]

The equation of motion of a particle in this field is

\[ \frac{dp}{dt} = \frac{Ze}{c} \vec{\nu} \times (\vec{B}_0 + \vec{B}_1) \]

In the reference frame of the waves, the momentum of the particle remains unchanged in module but changes in direction due to the perturbation:

\[ \frac{d\mu}{dt} = \frac{Zev(1 - \mu^2)^{1/2}}{pc} B_1 \cos[(\Omega - kv\mu)t + \psi] \]

\[ \Omega = \frac{ZeB_0}{mc\gamma} \]

\[ D(p) = \frac{1}{3} v\lambda \approx \frac{1}{3} \frac{cr_L(p)}{F(p)} \]

\[ F(p(k)) = k(\delta B / B)^2 \]

The Diffusion coeff reduces To the Bohm Diffusion for Strong Turbulence F(p)~1
In the LC approach the lowest diffusion coefficient, namely the highest energy, can be achieved when $F(p) \approx 1$ and the diffusion coefficient is Bohm-like.

\[ D(p) = \frac{1}{3} v_{\lambda} \approx \frac{1}{3} \frac{c r_L(p)}{F(p)} \]

\[ \tau_{\text{acc}} \approx \frac{D(E)}{u_{\text{shock}}^2} = 3.3 \times 10^6 \frac{E_{\text{GeV}}}{B_{\mu}^{-1}} u_{1000}^{-2} \text{ sec} \]

For a life-time of the source of the order of 1000 yr, we easily get $E_{\text{max}} \sim 10^{4-5} \text{ GeV}$

We recall that the knee in the CR spectrum is at $10^6 \text{ GeV}$ and the ankle at $\sim 3 \times 10^9 \text{ GeV}$. The problem of accelerating CR's to useful energies remains…

BUT what generates the necessary turbulence anyway?

Wave growth HERE IS THE CRUCIAL PART!

\[ \frac{\partial F}{\partial t} + u \frac{\partial F}{\partial x} = \sigma F - \Gamma F \]

Bell 1978
Standard calculation of the Streaming Instability (Achterberg 1983)

\[
\frac{c^2 k^2}{\omega^2} = 1 + \chi^{R,L}
\]

\[
\chi^{R,L} = \frac{4\pi^2 e^2}{\omega} \int dp \int d\mu \frac{p^2(1 - \mu^2)v^2(p)}{\omega + \Omega' - v\mu k} \left[ \frac{\partial f}{\partial p} + \frac{1}{p} \left( \frac{k\nu}{\omega} - \mu \right) \frac{\partial f}{\partial \mu} \right]
\]

There is a mode with an imaginary part of the frequency: CR's excite Alfven Waves resonantly and the growth rate is found to be:

\[
\sigma = V_A \frac{\partial P_{CR}}{\partial z}
\]
Maximum Level of Turbulent Self-Generated Field

\[ \frac{\partial F}{\partial t} + u \frac{\partial F}{\partial x} = \sigma F \]

Stationarity

\[ u \frac{\partial F}{\partial z} = v_A \frac{\partial P_{CR}}{\partial z} \]

Integrating

\[ \frac{\delta B^2}{B_0^2} = 2 MA \frac{P_{CR}}{\rho u^2} \gg 1 \]

Breaking of Linear Theory...

For typical parameters of a SNR one has $\delta B/B \sim 20$. 
We Generalized the previous formalism to include the Precursor!

We Solved the Equations for a CR Modified Shock together with the eq. for the self-generated Waves

We have for the first time a Diffusion Coefficient as an output of the calculation
Spectra of Accelerated Particles and Slopes as functions of momentum
Magnetic and CR Energy as functions of the Distance from the Shock Front

Amato & PB 2006
Super-Bohm Diffusion

Amato & PB 2006
How do we look for NL Effects in DSA?

- **Curvature in the radiation spectra (electrons in the field of protons)** – (indications of this in the IR-radio spectra of SNRs by Reynolds)

- **Amplification in the magnetic field at the shock** (seen in Chandra observations of the rims of SNRs shocks)

- **Heat Suppression downstream** (detection claimed by Hughes, Rakowski & Decourchelle 2000)

- All these elements are suggestive of very efficient CR acceleration in SNRs shocks. BUT similar effects may be expected in all accelerators with shock fronts
A few notes on NLDS

- The spectrum “observed” at the source through non-thermal radiation may not be the spectrum of CR’s.
- The spectrum at the source is most likely concave though a convolution with $p_{\text{max}}(t)$ has never been carried out.
- The spectrum seen outside (upstream infinity) is peculiar of NL-DSA:

\[
F(E) \quad E \quad p_{\text{max}}(t) \quad E
\]
CONCLUSIONS

- Particle Acceleration at shocks occurs in the non-linear regime: these are NOT just corrections, but rather the reason why the mechanism works in the first place.

- The efficiency of acceleration is quite high.

- Concave spectra, heating suppression and amplified magnetic fields are the main symptoms of NL-DSA.

- More work is needed to relate more strictly these complex calculations to the phenomenology of CR accelerators.

- The future developments will have to deal with the crucial issue of magnetic field amplification in the fully non-linear regime, and the generalization to relativistic shocks.
Exact Solution for Particle Acceleration at Modified Shocks
for Arbitrary Diffusion Coefficients

Amato & Blasi (2005)

Basic Equations

\[ \rho(x)u(x) = \rho_0 u_0 \]

\[ \rho(x)u(x)^2 + P_g(x) + P_c(x) = \rho_0 u_0^2 + P_{g,0} \]

\[ \frac{\partial f(p,x)}{\partial t} = \frac{\partial}{\partial x} \left[ D(p,x) \frac{\partial f(p,x)}{\partial x} \right] - u(x) \frac{\partial f(p,x)}{\partial x} + \frac{1}{3} \frac{du}{dx} p \frac{\partial f(p,x)}{\partial p} + Q(x,p,t) \]

\[ P_c(x) = \frac{4\pi}{3} \int dp p^3 \nu(p) f(p,x) \]

\[ f(x,p) \approx f_0(p) \exp \left[ -\frac{q(p)}{3} \int dx' \frac{u(x')}{D(x',p)} \right] \quad q(p) = -\frac{d\ln f_0}{d\ln p} \]

\[ f_0(p) = \left( \frac{3R_{tot}}{R_{tot} U(p) - 1} \right) \frac{\eta n_0}{4\pi p^3_{inj}} \exp \left[ -\int dp' \frac{3R_{tot} U(p')}{p' R_{tot} U(p') - 1} \right] \]

DISTRIBUTION FUNCTION AT THE SHOCK
It is useful to introduce the equations in dimensionless form:

\[ \xi_c(x) = 1 + \frac{1}{\gamma M_0^2} U(x) - \frac{1}{\gamma M_0^2} U(x)^{-\gamma} \quad \xi_c(x) = \frac{P_c(x)}{\rho_0 u_0^2} \]

\[ \xi_c(x) = \frac{4\pi}{3\rho_0 u_0^2} \int dp \int p^3 v(p) f_0(p) \exp \left[ - \int dx' \frac{U(x')}{x_p(x')} \right] \quad x_p(x) = \frac{3D(p, x)}{q(p) u_0} \]

One should not forget that the solution still depends on \( f_0 \) which in turn depends on the function

\[ U(p) = \int dx U(x)^2 \frac{1}{x_p(x)} \exp \left[ - \int dx' \frac{U(x')}{x_p(x')} \right] \]

Differentiating with respect to \( x \) we get ...
\[ \frac{d\xi_c}{dx} = -\lambda(x)\xi_c(x)U(x) \]

\[ \lambda(x) = \frac{\int dp \ p^3 \frac{1}{x_p(x)} v(p)f_0(p) \exp\left[-\int dx' \frac{U(x')}{x_p(x')}\right]}{\int dp \ p^3 v(p)f_0(p) \exp\left[-\int dx' \frac{U(x')}{x_p(x')}\right]} , \]

The function \( \xi_c(x) \) has always the right boundary conditions at the shock and at infinity but ONLY for the right solution the pressure at the shock is that obtained with the \( f_0 \) that is obtained iteratively (NON LINEARITY)