γ Ray Showers

86 – 88

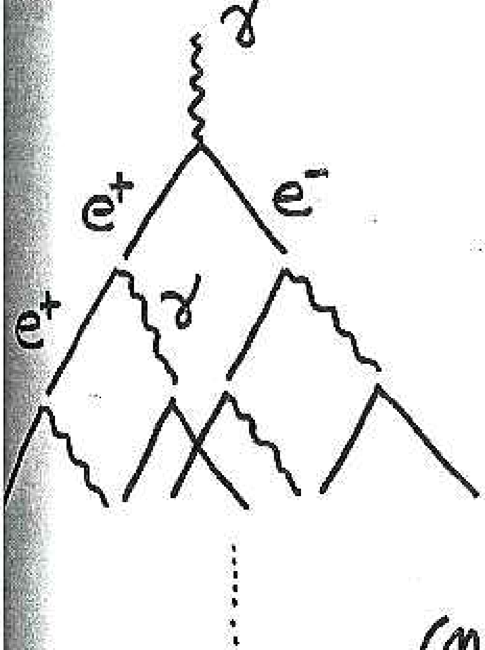
JPM3

NOTICE THAT

$$\frac{\xi_{pair}}{\xi_{bremss}} = \frac{\sigma_{bremss}}{\sigma_{pair}} \simeq \frac{\frac{28}{9}\ln\frac{2E}{m_e} - \frac{218}{7}}{4\left(\ln\frac{183}{Z^{1/3}} + \frac{1}{8}\right)}$$

$$\simeq 1 \text{ for } E \simeq 1\,\text{GeV} \quad Z = 7\text{-}8 \text{ (air)}$$

We can treat a high energy shower as if radiation and pair production occur after the same distance



		total # particles in layer
R	E_0	$= 1$
$2R$	$\frac{E_0}{2}$	~~3~~ 2
$3R$	$\frac{E_0}{4}$	~~4~~ 4
$4R$	$\frac{E_0}{8}$	~~5~~ 8
⋮	⋮	⋮
$(n \cancel{+1})R$	$\frac{E_0}{2^n}$	2^n

After $(n-1)$ interactions the energy of a particle (e or γ, roughly $\frac{2}{3}\, e^{\pm}$ and $\frac{1}{3}\,\gamma$) has degraded to $E_0/2^n \simeq E_{critical} \simeq m_e$; The shower reaches its "maximum". ($\rightarrow$ ctd

$$x = n\xi_0 \ln 2$$

$$n_{max} = \frac{E_0}{E_c}$$

$$\int dx\, E \frac{dN}{dE\,dx} = \int^{n_{max}} \underbrace{2^n}_{y} \underbrace{dn}_{\frac{1}{\ln 2}\frac{dy}{y}}$$

$$2^n = y$$

$$n \ln 2 = \ln y$$

$$dn = \frac{1}{\ln 2} \frac{dy}{y}$$

$$= \frac{1}{\ln 2} n_{max} = \frac{1}{\ln 2} \frac{E_0}{E_c}$$

$$\frac{dN}{dE} = \left(\frac{E_0}{\ln 2\, E_c}\right) \frac{1}{E}$$

It is absorbed by the atmosphere. $e^\pm$'s ionize atoms and γ's are lossed by photoelectric absorption.

Properties of the shower

- R is the distance after which the γ has lost 1/2 of its primary energy E_0

$$e^{-\frac{R}{\xi_0}} = \frac{1}{2} \rightarrow \quad R = \xi_0 \ln 2$$

- at shower max

$$\frac{E_0}{2^{n_{max}}} = E_c \rightarrow n_{max} = \frac{\ln(E_0/E_c)}{\ln 2}$$

- number of particles at max ~ energy E_c

$$n = 2^{(n_{max})} = \left(\frac{E_0}{E_c}\right)$$

- exponential growth to n_{max} radiation length followed by rapid attenuation

eg

$E = 100$ GeV
$E_c \sim 1$ MeV

$\frac{E}{E_c} = 10^5$

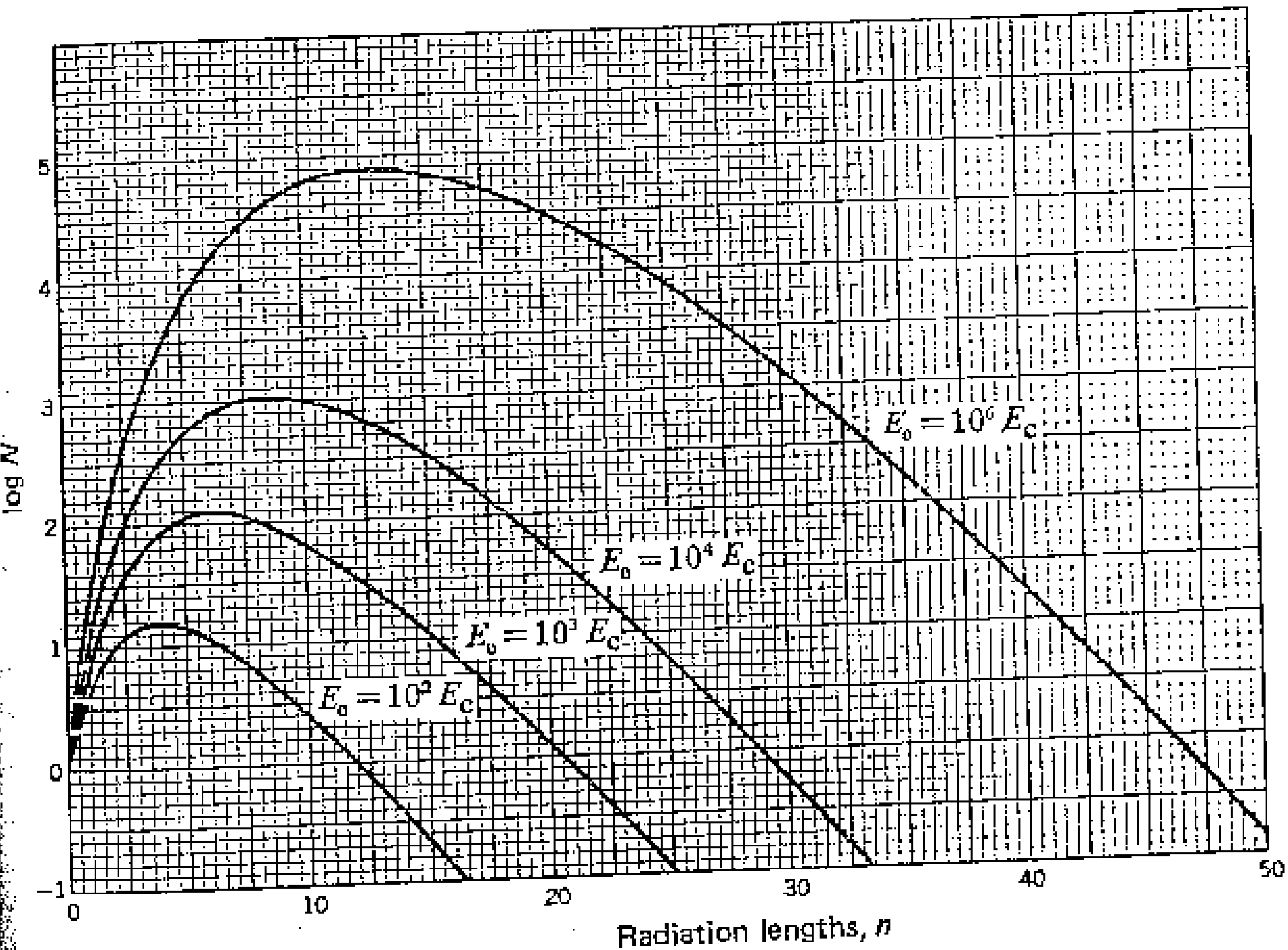
$n_{max} = \frac{\ln 10^5}{\ln 2} = \frac{11.5}{0.7}$

$n = 2 \times 10^5$

spectrum $\quad n(E,R) = 2^R$

$$\int_0^{R_{max} = \frac{\ln(E_0/E)}{\ln 2}} 2^R \, dR = \frac{E_0}{\ln 2} \frac{1}{E}$$

Fig. 4.6. The total number of particles N in a shower initiated by an electron of energy E_0, as a function of depth n, measured in radiation lengths; E_c is the critical energy of the material. (From Leighton, 1959, p. 693, after Rossi & Greisen, 1941.)



Cascade Equations

One of the most important techniques in high energy astrophysics. E.g.

- the problem of a particle beam (from pulsar) propagating through companion atmosphere
- particles propagating out from the sun's interior
- propagation of particles through ISM
- propagation of CR through atmosphere

...

We will discuss the last problem but the techniques are the same for all other problems

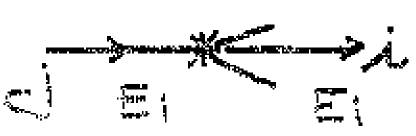
i disappears by interaction

i disappears by decay

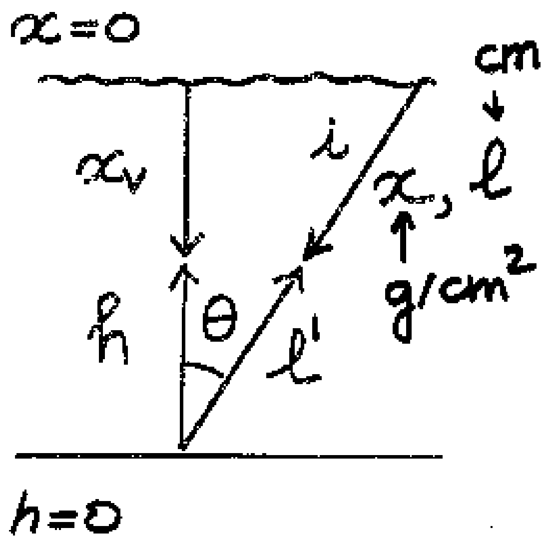
$$\frac{dN_i(E,x)}{dx} = \ominus \frac{1}{\lambda_i} N_i(E,x) \ominus \frac{1}{d_i} N_i(E,x)$$

$$\oplus \sum_j \int \frac{F_{ji}(E_i, E_j)}{E_i} \frac{N_j(E_j)}{\lambda_j} dE_j$$

i is produced in interaction of j on N, O



Symbols in the cascade equation



= height

= zenith angle

$\tau = \gamma c t$ $\qquad d_i = \gamma c \tau_i \rho$

$$\frac{\Delta N_i}{N_i} = \frac{\Delta d}{\gamma c \tau_i} = \frac{\Delta x}{\rho \gamma c \tau_i} = -\frac{\Delta x}{d_i}$$

ling

$$F_{ji} = E_i \frac{dN_i}{dE_i}(E_i, E_j)$$

$$\Downarrow$$

$$= \frac{E_i}{E_j} \frac{dN_i}{d(E_i/E_j)}\left(\frac{E_i}{E_j}\right)$$

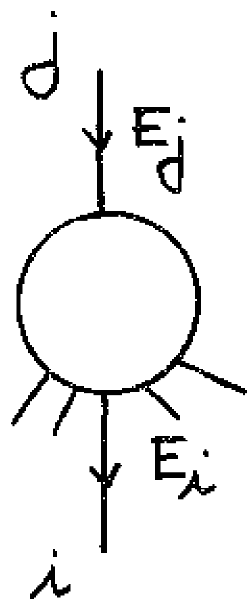
$$= \frac{1}{\sigma_j} x_i \frac{d\sigma_i}{dx_i}(x)$$

→ $N_i(E, x)$: number of particles i at a depth x (in g/cm²) in the atmosphere of energy E

→ λ_i : average depth at which the particles interact (interaction or radiation length; in g cm⁻²) (see 83)

→ d_i : average decay length if i is unstable ($d_i = \rho \gamma c \tau_i$)

→ F_{ji} : dimensionless cross section for a particle j of energy E_j to produce ~~a~~ particle of type i and energy E_i in an interaction with an air nucleus



- total cross section of particle j on air is σ_j
- $d\sigma_i/dx$ is the inclusive cross section for j to produce i with a fraction $x = E_i/E_j$ of its original momentum

measured by accelerators

Notice ($E_{i,j} \gg m_{i,j}$)

$$\rightarrow \quad 0 \leq x_L \left(= \frac{E_i}{E_j}\right) \leq 1$$

$$\rightarrow \quad \int_0^1 dx_L \left[\frac{1}{\sigma_j} \frac{d\sigma_i(x_L)}{dx_L}\right] = \langle n_i \rangle$$

$$\rightarrow \quad \int_0^1 dx_L \, x_L \left[\frac{1}{\sigma_j} \frac{d\sigma_i(x_L)}{dx_L}\right] = \langle n_i x_i \rangle$$

$\langle n_i \rangle$, $\langle x_i \rangle$ is the average number (multiplicity) and the average (relative) momentum of particles i produced in a air collision.

Cascade Equations in Approximation A

Assumptions:

1. scaling: $F_{ji}(E_i, E_j)$ depends on $x_L = E_i / E_j$ only
2. λ independent of E, i.e. cross section vary weakly with energy

The cascade eq of (110) can now be written

$$\frac{dN_i(E,x)}{dx} = -\left(\frac{1}{\lambda_i} + \frac{1}{d_i}\right) N_i(E,x)$$

$$+ \frac{1}{\lambda_j} \int_0^1 \frac{dx_L}{x_L^2} \Big[\underbrace{\frac{x_L}{\sigma_j} \frac{d\sigma}{dx_L}(x_L)}_{\text{see (111)}} \Big] N_j\left(\frac{E}{x_L}\right)$$

Derivation:

$$x_L = \frac{E_i}{E_j} \equiv \frac{E}{E_j}$$

$$\frac{dE_j}{E_i} = d\left(\frac{1}{x_L}\right) = -\frac{dx_L}{x_L^2}$$

$$E_j = \frac{E_i}{x_L} \equiv \frac{E}{x_L}$$

Nucleons in the atmospheric cascades (or elementary solutions of the cascade eq.)

$N(E,x)$ flux of nucleons

$$\frac{dN}{dx}(E,x) = -\frac{1}{\lambda_N} N(E,x) + \int_0^1 \frac{dx_L}{x_L^2} \cdot \frac{N(E/x_L)}{\lambda_N} \left[\frac{x_L}{\sigma_N} \frac{d\sigma_{NN}(x_L)}{dx_L} \right]$$

λ_N : interaction length of nucleons in air.
(defined as radiation length, see (78))

$$\lambda_N = \frac{\rho(\text{air})}{\rho(\text{nucleons})} \frac{1}{\sigma_N^{air}} = 80 \frac{g}{cm^2}$$

$\rho(\text{nucleons}) \rightarrow A\, m_N$, $\sigma_N^{air} \simeq 300\,mb$

A: 14.4 averaged over N, O

$$\lambda_N = \frac{A\, m_N}{\sigma_N^{air}} \qquad \sigma_N^{air} \simeq (14.4)^{2/3} \sigma_{NN}$$

σ_{NN}: 50 mb

NN total cross section

Nucleons (ctd) <u>155</u>

The nucleon cascade eq (114) has to be solved with the boundary condition at the top of the atmosphere (x=0)

- $N(E,0) = A\,\delta\left(E - \frac{E_0}{A}\right)$ i.e. nucleus (with A nucleons) of primary energy E_0

depends of energy region fitted (composition!)

- $N(E,0) = \frac{dN}{dE} = 1.8\, E^{-2.7}$ $(E \lesssim 10^3\ \mathrm{TeV})$

$\left(\text{in } \frac{\text{nucleons}}{\mathrm{cm^2\,s\,sr\,GeV/A}}\right)$

i.e. cosmic ray flux

Elementary solution

$$\boxed{N(E,x) = G(E)\,g(x)}$$

energy & depth dependence factorize

Substitute in (114)

$$G\,dg/dx = -\frac{Gg}{\lambda_N} + g\int_0^1 \frac{dx_L}{x_L^2}\,\frac{G(E/x_L)}{\lambda_N}\left[\frac{x_L}{\sigma_N}\frac{d\sigma_{Ni}}{dx_L}\right.$$

or

$$\underbrace{\frac{dg/dx}{g} = -\frac{1}{\Lambda}}\ ;\quad \boxed{\frac{1}{\Lambda} = \frac{1}{\lambda_N} - \frac{1}{G(E)}\int_0^1 \frac{dx_L}{x_L^2}\ \ldots\ldots}$$

$$\boxed{g(x) = g(0)\,e^{-\frac{x}{\Lambda}}}$$

Nucleons (ctd)

use previous solution with the boundary condition

$$G(E) \sim E^{-(\gamma+1)} \qquad (\gamma = 1.7 \text{ for CR})$$

$$\Rightarrow \boxed{N(E,x) = g(0)\, e^{-\frac{x}{\Lambda_N}} E^{-(\gamma+1)}}$$

$$\frac{1}{\Lambda_N} = \frac{1}{\lambda_N}\left[1 - \int_0^1 \frac{dx_L}{x_L^2} \underbrace{\frac{G(E/x_L)}{G(E)}}_{x_L^{\gamma+1}} \left[\frac{x_L}{\sigma_N}\frac{d\sigma_{NN}}{dx_L}\right]\right]$$

$$\Rightarrow \boxed{\frac{1}{\Lambda_N} = \frac{1}{\lambda_N}\left[1 - \underbrace{\int_0^1 dx_L\, x_L^{\gamma-1}\left[\frac{x_L}{\sigma_N}\frac{d\sigma_{NN}}{dx_L}\right]}_{\equiv z_{NN}}\right]}$$

Discussion:

Above equation describe the flux of nucleons in the atmosphere initiated by CR bombarding the top. Both the energy and depth dependence of the number of nucleons is given. For a $\gamma = 1$ (normal) spectrum z_{NN} is just the average relative momentum $\langle x_L \rangle$ of nucleons produced in N-air collisions (see (112). For $\gamma > 1$ (e.g. 1.7) the production of $x_L \simeq 0$ N's is suppressed in Λ. The production of high E N's is important as the push the casca

Pions and Kaons in Atmospheric Cascades

We discuss π's. K's can be treated in identical fashion. The treatment of π's is more complicated as

i) they decay (this will be very important later as this decay is the primary source of CR muons and neutrinos $\pi^{\pm} \to \mu\nu$

ii) as nucleons they interact to produce more π's

iii) π's are also produced by nucleons

Cascade eq. for pion flux $\pi(x, E)$

$$\pi(E, x=0) = 0$$

$$\frac{d\pi}{dx}(E,x) = \left(-\frac{1}{\lambda_\pi} - \frac{1}{d_\pi}\right)\pi$$

$$+ \frac{\pi}{\lambda_\pi}\int_0^1 dx_L\, x_L^{\gamma-1}\left[\frac{x_L}{\sigma_\pi}\frac{d\sigma_{\pi\pi}}{dx_L}\right]$$

$$+ \frac{N}{\lambda_N}\int_0^1 dx_L\, x_L^{\gamma-1}\left[\frac{x_L}{\sigma_N}\frac{d\sigma_{N\pi}}{dx_L}\right]$$

$$\pi' = -\frac{\pi}{\lambda_\pi} - \frac{\pi}{d_\pi} + \frac{\pi}{\lambda_\pi} Z_{\pi\pi} + \frac{N}{\lambda_N} Z_{N\pi}$$

π's and K's (ctd)

e.g. let us take the high E limit. High energy π's have an increased lifetime, so they travel distances $\gamma c \tau$ large compared to the atmosphere ($\simeq 10$ km). We can neglect π/d_π term and (117) can be written

$$\Rightarrow \boxed{\frac{d\pi}{dx} = -\frac{\pi}{\Lambda_\pi} + \frac{N}{\lambda_N} Z_{N\pi}} = -\frac{\pi}{\Lambda_\pi} + a e^{-\frac{x}{\Lambda_N}}$$

$$a = N_0 \frac{Z_{N\pi}}{\lambda_N}$$

$$\begin{cases} \Lambda_\pi^{-1} = \lambda_\pi^{-1}(1 - Z_{\pi\pi}) \\ N = N_0(E)\, e^{-\frac{x}{\Lambda_N}} \quad (\text{see } 116) \end{cases}$$

$$N_0(E) = g(0) E^{-(\gamma+1)} \quad \text{and} \quad \Lambda_N^{-1} = \lambda_N^{-1}(1 - Z_{NN})$$

Solution:

$$\pi = \underbrace{e^{-\frac{x}{\Lambda_\pi}}}\, a \underbrace{\int_0^{x} dx'\, e^{\left(\frac{x'}{\Lambda_\pi} - \frac{x'}{\Lambda_N}\right)}}$$

solution!

Substitute two factors

$$\frac{d\pi}{dx} = -\frac{1}{\Lambda_\pi}\pi + e^{-\frac{x}{\Lambda_\pi}} a \left(e^{\frac{x}{\Lambda_\pi}} e^{-\frac{x}{\Lambda_N}} \right) \checkmark$$

$$\pi = a e^{-\frac{x}{\Lambda_\pi}} \left(\frac{1}{\Lambda_\pi} - \frac{1}{\Lambda_N}\right)^{-1} \left[e^{\frac{x}{\Lambda_\pi}} e^{-\frac{x}{\Lambda_N}} - 1 \right]$$

$$\pi = N_0 \frac{Z_{N\pi}}{\lambda_N} \frac{\Lambda_\pi \Lambda_N}{-\Lambda_N + \Lambda_\pi} \left[e^{-\frac{x}{\Lambda_\pi}} - e^{-\frac{x}{\Lambda_N}} \right].$$

$$\boxed{\pi = N \frac{Z_{N\pi}}{\quad} \frac{\Lambda_\pi}{\quad} \left[e^{-\frac{x}{\Lambda_\pi}} - e^{-\frac{x}{\Lambda_N}} \right]}$$

INPUT : from accelerator data

(all λ, Λ 's in grams cm^{-2}) (for air)

$$\left.\begin{array}{l} Z_{pp} = 0.27 \\ Z_{pn} = 0.034 \end{array}\right\} \; Z_{NN} \simeq 0.30 \simeq Z_{\pi\pi}$$

$$\left.\begin{array}{l} Z_{p\pi^+} = 0.046 \\ Z_{p\pi^-} = 0.035 \end{array}\right\} \; Z_{N\pi} \simeq 0.081$$

$Z_{p\pi^0} = 0.041$

$Z_{pK^+} = 0.0092$

$Z_{pK^-} = 0.0030$

$Z_{p(K^0+\bar{K}^0)} = 2 \; Z_{pK^-} = 0.0060$

$\lambda_N = 86$ $\qquad$ $\lambda_\pi = 1.3 \, \lambda_N$

$\Lambda_N = 123$

$\Lambda_\pi = 162$ $\qquad$ $\Lambda_i = \dfrac{\lambda_i}{1 - Z_{ji}}$

$$\frac{1.3 \times 86}{1 - 0.30} =$$

Muons in Air Showers.

Low energy limit
all π decay
without interaction

Origin : $\pi \to \mu\nu$ decay

$$\frac{d\pi}{dx} = -\left(\frac{1}{\Lambda_\pi} + \frac{1}{d_\pi(E)}\right)\pi + \frac{N_0(E)\, Z_{N\pi}\, e^{-\frac{x}{\Lambda_N}}}{\lambda_N}$$

(neglects $\pi \to \pi \to$ decay)

$$\Lambda_\pi = \frac{\lambda_\pi}{1 - Z_{\pi\pi}}$$

π-decay

$$t_\pi = \frac{\tau_\pi}{1 - v^2/c^2} = \gamma \tau_\pi$$

$$\frac{d\pi}{\pi} = -\frac{dl}{\underset{\overset{\parallel}{c}}{v}\, t_\pi} = -\frac{dl}{\gamma c \tau_\pi}$$

(see Fig III for notation)

atmosphere $\quad dx = \rho\, dl$

$$\boxed{\begin{aligned} x_v &= x_0 e^{-h_v/h_0} \\ h_v &= l\cos\theta \\ x_0 &\simeq 1030 \text{ g/cm}^2 \\ h_0 &\simeq 6.4 \text{ km (at sea level)} \end{aligned}}$$

really a function of height

decay length

g/cm^2 → $d_\pi = \rho(\gamma c \tau_\pi)$ $dx = \rho\, dl$

for atmosphere

$$\rho = -\frac{dx}{dl} = -\frac{1}{\cos\theta}\frac{dx_v}{dl} = -\frac{x_0}{\cos\theta}\frac{d}{dl}e^{-\frac{l\cos\theta}{h_0}}$$

$$\rho = \underbrace{\frac{x_0}{\cos\theta}\frac{\cos\theta}{h_0}e^{-\frac{l\cos\theta}{h_0}}}_{x} = \frac{x\cos\theta}{h_0}$$

$$\therefore\ d_\pi = \frac{x\cos\theta}{h_0}\cdot\frac{E}{m_\pi c^2}c\tau_\pi \equiv \frac{Ex\cos\theta}{\epsilon_\pi}$$

$d_\pi \sim$ lifetime $\sim$ energy E →

$$\boxed{d_\pi = \frac{Ex\cos\theta}{\epsilon_\pi} \qquad \epsilon_\pi = \frac{h_0}{c}\frac{(m_\pi c^2)}{\tau_\pi}}$$

→ favors large zenith angles

→ ! (long d_π in rare atmosphere)

← characteristic decay energy

high energy π's live longer, d_π longer, less likely to decay (than interact)

ϵ_π : energy of close competition between decay and interaction

Particle	ϵ (GeV)
μ	1
$\pi^{\pm}$	115
π^{0}	3.5×10^{10}
$K^{\pm}$	850
K_S	1.2×10^{5}
K_L	205

π-cascade : solution

from 120

$$\rightarrow \quad \pi' = -\left(\frac{1}{\Lambda_\pi} + \frac{\delta}{x}\right) + \eta e^{-\frac{x}{\Lambda_N}}$$

(neglects $\pi \to \pi$)

$$\delta(E) \equiv \frac{x}{d_\pi} \qquad \delta \equiv \frac{\epsilon_\pi}{E\cos\theta}$$

$$\eta(E) \equiv \frac{N_0(E)\, z_{N\pi}}{\lambda_N}$$

high energy (compared to ϵ_π) is δ small $\rightarrow \left(\frac{x'}{x}\right)^{\delta} = 1$ Do exp integral

Solution

$$\pi = \eta e^{-\frac{x}{\Lambda_\pi}} \int_0^{x} e^{\frac{x'}{\Lambda_\pi}} e^{-\frac{x'}{\Lambda_N}} \left(\frac{x'}{x}\right)^{\delta} dx'$$

Proof: by calculating $d\pi/dx$ relative to factors circled

Muons (ctd)

$$-\delta\left(\frac{x'}{x}\right)^{\delta-1}\left(-\frac{x'}{x^2}\right)$$

$$\frac{d}{dx}\left(\frac{x'}{x}\right)^{\delta}$$

✓

$$\pi' = -\frac{\pi}{\Lambda_\pi} + \eta\, \cancel{e^{-\frac{x}{\Lambda_\pi}}}\, \cancel{e^{\frac{x}{\Lambda_\pi}}}\, e^{-\frac{x}{\Lambda_N}} \cancel{(1)^{\delta}}$$

✓

$$+\underbrace{\eta\, e^{-\frac{x}{\Lambda_\pi}} \int_0^x dx'\, e^{-\frac{x'}{\Lambda_\pi}} e^{-\frac{x'}{\Lambda_N}} (-\delta x^{-\delta} x^{-1}) x'^{\delta}}_{-\frac{\delta}{x}\pi \ ✓}$$

Other form:

$$\pi(E,x) = \int_0^x \frac{dn_\pi(E,x')}{dx'} P_s(x-x')\, dx'$$

$$P_s(x-x') = \left(\frac{x'}{x}\right)^{\delta} e^{-\left(\frac{x-x'}{\lambda_\pi}\right)}$$

P_s : probability that a π produced at x' survives to depth x.

High Energy limit : $d_\pi \to \infty$, $\frac{1}{d_\pi} \to 0$ no decay

Low Energy limit : $\underbrace{\delta \text{ very large}}$, all π's decay

$$\left(\frac{x'}{x}\right)^{\delta} \to 0 \text{ as } \frac{x'}{x} < 1$$

except for $x' = x$

$$\pi(E,x) = \eta\, e^{-\frac{x}{\Lambda_\pi}} \cancel{e^{\frac{x}{\Lambda_\pi}}}\, \cancel{e^{-\frac{x}{\Lambda_N}}} \int_0^x \left(\frac{x'}{x}\right)^{\delta} dx'$$

$$\frac{x}{\delta+1}\left(\frac{x'}{x}\right)^{\delta+1}\Bigg|_0^x = \frac{x}{\delta+1} \simeq \frac{x}{\delta} = d_\pi$$

Muons (ctd)

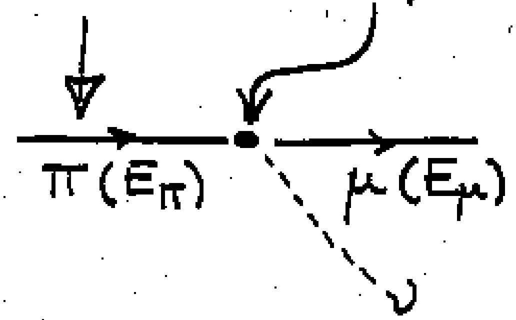
solution when all π's decay

$$\pi(E,x) = d_\pi \, \eta \, e^{-\frac{x}{\lambda_N}}$$

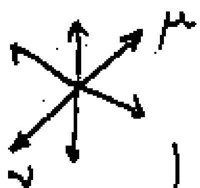
$$\boxed{\pi(E,x) = d_\pi \frac{N_0 Z_{N\pi}}{\lambda_N} e^{-\frac{x}{\lambda_N}}}$$

π decay → μν

$$\frac{dP_\mu}{dx}(E_\mu) = \int_{E_{min}}^{E_{max}} \frac{\pi(E_\pi, x)}{d_\pi} \frac{dN_{\pi\to\mu}}{dE_\mu}(E_\mu, E_\pi)\, dE_\pi$$

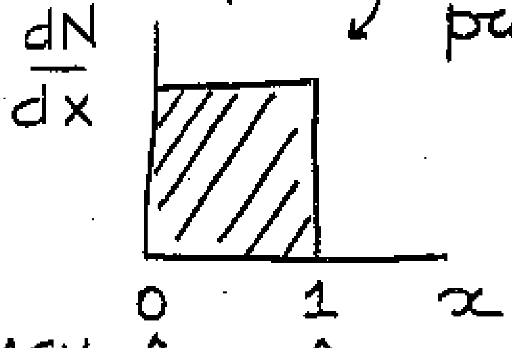


$\frac{dN_{\pi\to\mu}}{dE_\mu}$ from 2-body decay



uniform ang. distribution

projection



$x = E_\mu / E_\pi$

all energy in ν

all energy in μ

Muons (ctd)

$$B(\pi \to \mu\nu) = \int_0^1 \frac{dN}{dx}\, dx = \text{cte}$$

$$\therefore \quad \frac{dN}{dx} = B(\pi \to \mu\nu) \simeq 1$$

$x = E_\mu / E_\pi$

$$\frac{dN}{dE_\mu} = E_\pi^{-1} B(\pi \to \mu\nu) \simeq E_\pi^{-1}$$

This assumes $m_\mu = 0$, correct answer

$$\boxed{\frac{dN_{\pi\to\mu}}{dE_\mu} = \frac{1}{\xi_\pi E_\pi} \qquad \xi_\pi = 1 - \frac{m_\mu^2}{m_\pi^2}}$$

$\to \infty$ for $\xi \to 1$

from (124)

$$\boxed{\frac{dP_\mu}{dx}(E_\mu, x) = \int_{E_\mu}^{\frac{1}{1-\xi} E_\mu} \frac{\pi(E_\pi, x)}{d_\pi} \frac{1}{\xi_\pi E_\pi}\, dE_\pi}$$

Muons (ctd)

Use (125) in conjunction with $\pi(E,X)$ of (124). i.e. muons spectrum when all π's decay

$$\frac{dP_\mu}{dx} = \int_{E_\mu}^{\frac{1}{1-\xi}E_\mu} dE_\pi \, \frac{1}{d_\pi} \underbrace{\left(d_\pi \, \eta \, e^{-\frac{x}{\Lambda_N}}\right)} \frac{1}{\xi_\pi} \frac{1}{E_\pi}$$

$$z = \frac{E_\pi}{E_\mu} \left(= \frac{1}{x}\right)$$

$$\frac{dP_\mu}{dx} = \frac{1}{\xi} \int_1^{(1-\xi)^{-1}} \frac{\pi(zE_\mu, x)}{d_\pi} \frac{dz}{z}$$

$$= \frac{Z_{N\pi}}{\xi \lambda_N} e^{-\frac{x}{\Lambda_N}} \int_1^{(1-\xi)^{-1}} N_0(zE_\mu) \frac{dz}{z}$$

$$N_0(zE_\mu) = N_0(E_\mu) \, z^{-(\gamma+1)}$$

$$= \frac{N_0 \, Z_{N\pi}}{\xi \, \lambda_N} e^{-\frac{x}{\Lambda_N}} \underbrace{\int_1^{(1-\xi)^{-1}} z^{-(\gamma+2)} \, dz}_{\frac{1}{\gamma+1}\left[1-(1-\xi)^{\gamma+1}\right]}$$

int. over x

$$\boxed{P_\mu(E_\mu, x) = \frac{N_0(E_\mu) \, Z_{N\pi}}{\xi} \frac{\Lambda_N}{\lambda_N} \frac{1}{\gamma+1} \left[1-(1-\xi)^{\gamma+1}\right]}$$