Degeneracy pressure of electrons: Chandrasekhar mass

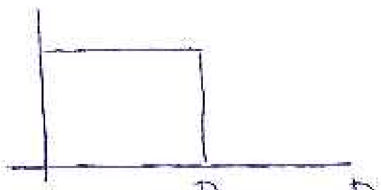
- pressure after nuclear burning
- identical particles: only $(2s+1)$ electrons can occupy an elementary cell in phase space $dx\,dy\,dz\,dp_x\,dp_y\,dp_z$ of size h^3

$$\boxed{\#\text{ states per unit volume} = \frac{d\left(\frac{4}{3}\pi p^3\right)}{h^3}}$$

number of states that can be squeezed in a volume V

$$N = (2s+1)\, V \int_0^{p_F} \frac{4\pi p^2 dp}{h^3} = 2\, V\, \frac{4\pi p_F^3}{3h^3}$$

can put e^-, e^+ in h^3



$$N/V = n_e$$

$$p_F = h\left[\frac{3n_e}{8\pi}\right]^{1/3}$$

non-relativistic / relativistic

$$P_{NR} = \frac{2}{3}\frac{E_{NR}}{V}$$

$$\boxed{\frac{E_{NR}}{V} = 2\int \frac{4\pi p^2 dp}{h^3}\left(\frac{p^2}{2m_e}\right) = \frac{8\pi p_F^5}{10\, m_e h^3} \sim n_e^{5/3}}$$

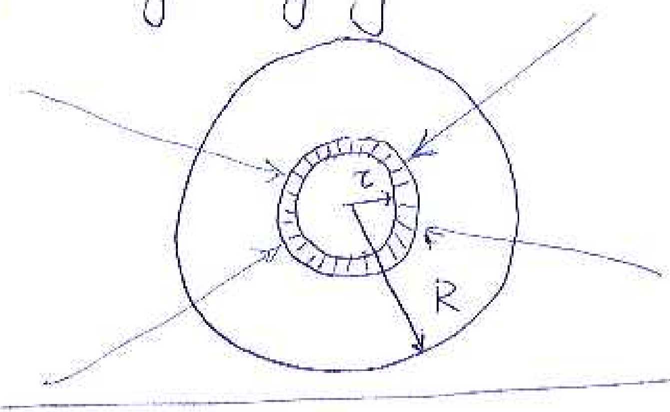
$$P_R = \frac{1}{3}\frac{E_R}{V}$$

$$\frac{E_R}{V} = \int \frac{8\pi p^2 dp}{h^3}\, pc = \frac{2\pi c}{h^3} p_F^4 \quad \propto n_e^{\frac{4}{3}}$$

Potential energy of a sphere of mass M and radius R

$$E_{Grav} = G\int_0^R dr \left[\left(\frac{4}{3}\pi r^3 \rho\right)\right]\left[\frac{1}{r}\right]\left[4\pi r^2 dr\, \rho\right] = \frac{3}{5} G\frac{M^2}{R}$$

assemble the sphere by bringing thin shells from ∞.



$$\boxed{P_{grav} = \frac{1}{3}\frac{E_{grav}}{V} = \frac{1}{5} G \left(\frac{4\pi}{3}\right)^{1/3} M^{2/3} \rho^{4/3}}$$ after eliminating R

ρ is mostly protons (neglect m_e). We can write it in terms of the electron # density n

$$n_e = \frac{Z}{A}\frac{\rho}{m_p}$$

So $P_{grav} \propto n_e^{4/3}$

- $\boxed{P_{NR} \propto n^{5/3}}$ $\quad P_R \propto n^{4/3}$ $\quad \boxed{P_{grav} \propto n^{4/3}}$

low mass stars: deg. pressure will support gravity
for suitable n_e

unstable: further collapse possible

- NR means $\quad p_F = h\left[\frac{3n}{8\pi}\right]^{1/3} < m_e c$

$$n_e^{-\frac{1}{3}} > \underbrace{\left(\frac{3}{8\pi}\right)^{1/3}}_{1/2} \underbrace{\left(\frac{h}{m_e c}\right)}_{\text{Compton wavelength}}$$

particles separated by > Compton wavel.

Critical number ~~mass~~ density of $\boxed{\text{nucleons}}$

$\cancel{\rho_0 = \frac{Z}{A}\frac{\rho_{N0}}{m_p}}$

$$\boxed{\rho_0 = \frac{A}{Z} m_p n_e = \frac{8\pi m_p A}{3Z}\left(\frac{m_e c}{h}\right)^3}$$

$\boxed{\text{Chandra mass}}$ $\quad \boxed{P_{NR} = P_{grav} \text{ with } \rho = \rho_0}$

$$P_{NR} = \left(\frac{3}{8\pi}\right)^{\frac{2}{3}} \left(\frac{h^2}{5m_e}\right) n_e^{5/3} = \left(\frac{G}{5}\right)\left(\frac{4\pi}{3}\right)^{1/3} M^{2/3} \left(\frac{m_p A}{Z}\right)^{4/3} n_e^{4/3}$$

$$\Big\downarrow \text{ solve for } n_e$$

$$\rho = \frac{A}{Z}\, m_p\, n_e = \left(\frac{4m_e^3 G^3 M^2}{h^6}\right)\left(\frac{Am_p}{Z}\right)^5\left(\frac{4\pi}{3}\right)^3$$

$$\rho = \rho_0 \longrightarrow \boxed{M_{Ch} = \left(\frac{3\sqrt{2}}{8\pi}\right)\left(\frac{hc}{G}\right)^{3/2}\left(\frac{Z}{Am_p}\right)^2 =}$$

$$= 4.91\left(\frac{Z}{A}\right)^2 M_\odot \simeq 1.2\, M_\odot$$

Many correction, e.g. charge of particles, ...

$$\boxed{M_{\text{Chandra}} = 1.2 - 1.8\, M_\odot}$$

Summary: • fill a sphere with electrons of Compton wavelength

$$\lambda = \frac{h}{m_e c} \qquad \rho_e = \frac{Z}{A}\frac{\rho_0}{m_p} = \frac{(25H)}{V}\underbrace{\left(\frac{V}{\frac{4\pi}{3}\left(\frac{h}{m_e c}\right)^3}\right)}_{N_e}$$

• calculate the NR pressure

• balance it with gravity

Galactic Supernova Explosions

- in last 1000 years

  1006 (Bayeux tapystry)
  1054 (Crab)
  1572 (Tycho)
  1604 (Kepler)
  ~1600 (casA, not seen visually, at 3kpc brightest radio source in the Galaxy)

- estimate of actual number

$$\frac{\text{unseen}}{\text{seen}} \simeq \frac{5}{1000\,\text{yrs}} \times \left(\frac{10\,\text{kpc}}{3\,\text{kpc}}\right)^2 \simeq \frac{1}{20\,\text{years}}$$

  this is consistent with
  → pulsar birth rate (though only Crab has a pulsar)
  → death of massive stars ($\gtrsim 10 M_\odot$)
  → abundance of heavy elements in Galaxy

- shell travels through interstellar medium with $v \gtrsim 200000 \frac{km}{s}$ to $150 \frac{km}{s}$ depending on age.

- type I: further collapse of e^- degeneracy caused by accreting companion star

Comment: neutron star as a nucleus of atomic# A

radius of a nucleus $R = 1.2\,\text{fm}\ A^{1/3}$

$$R = 15\ \text{km} \quad \text{for} \quad A = \frac{1.5\,M_\odot}{m_p} = 2\times10^{57}$$

$$\rho = \frac{1.5\,M_\odot}{\frac{4}{3}\pi R^3} = 2\times10^{17}\ \text{kg}\,\text{m}^{-3}$$

100 MeV per nucleon

$$E_{grav} = \frac{3}{5}\,G\,\frac{M^2}{R} = \frac{3}{5}\,G\,\frac{[1.5\,m_p A]^2}{1.2\,\text{fm}\ A^{1/3}} = 2\times10^{59}\ \text{MeV}$$

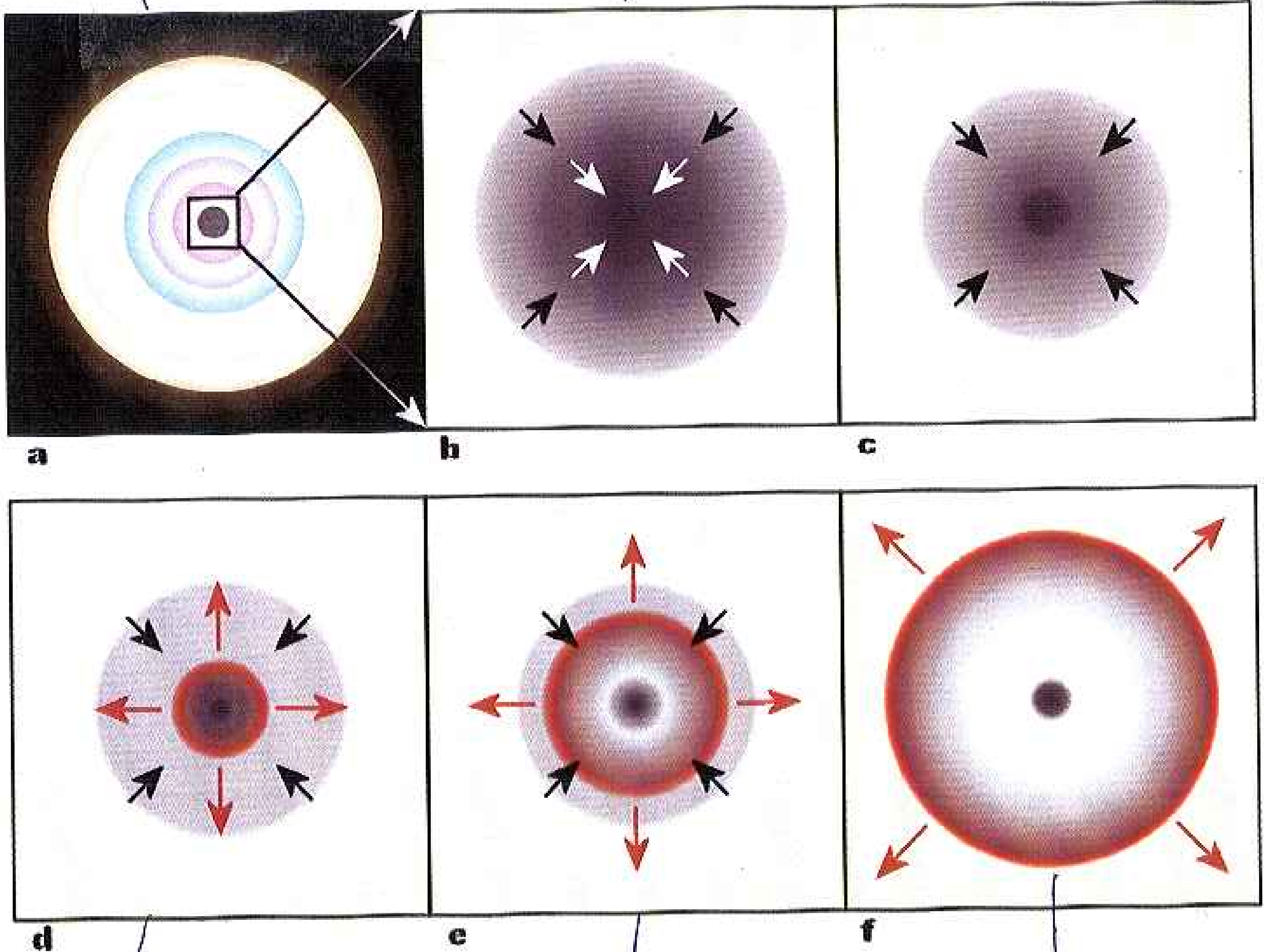
This 100 MeV is carried off by 6 neutrinos ($\Rightarrow$ 15 MeV)
These are the only particles to escape the ν-sphere

Neutrinos from Supernova Collapse

H He C Ne O Si Fe
core

Fe burning stops
→ collapse to $M_{Chandra}$
→ $e^- + p \to n + \nu_e$
from degenerate core

formation of degenerate
n-star with $M = 1.2 - 1.8\, M_\odot$
$R \cong 1.2\,fm\, A^{1/3} \cong 15\, km$
$L \;\; \frac{1.5\, M_\odot}{m_p} \simeq 2 \times 10^{57}$ of 100 MeV
$\rho = 2 \times 10^{17}\, kg/m^3$



binding energy $\frac{3}{5}\frac{M^2}{R} = E_{grav}$ released, $f_B \simeq 10\%$ in shock and $1 - f_B$ in neutrinos

outgoing shock with $f_B\, E_{grav}$ energy.

neutrino sphere form with $(1 - f_N)\, E_{grav}$ in ν's in equilibrium
$\gamma \leftrightarrow e^+ + e^- \leftrightarrow \nu + \bar{\nu}$

$\boxed{\text{Neutrino Sphere}}$

$$\gamma \leftrightarrow e^+ + e^- \leftrightarrow \nu_i + \bar{\nu}_i \qquad i = e, \mu, \tau$$

$\gamma, e^\pm, \nu_i$ reach thermal equilibrium with neutron & protons

- neutrinos with $\frac{1}{n\sigma_\nu} = \lambda_\nu < R \simeq 15\,\text{km}$ are trapped

$$\sigma(\nu_e + n \to p + e^-) = \sigma_\nu = \frac{G_F^2}{\pi} \frac{(kT_\nu)^2}{(\hbar c)^4} \simeq \frac{1}{2}\sigma_0 E_\nu^2 \quad (\sigma_0 = 1.8 \times 10^{-44}\,\text{cm}^2)$$

$E_\nu = 3.15\, T_\nu$ for Fermi distribution

$$n = \frac{\rho}{m_N} \sim 2\times 10^{17}\,\text{kg}\,\text{m}^{-3}$$

$$\boxed{\lambda \simeq \frac{900\,\text{m}}{E_{\text{MeV}}^2}}$$

$\boxed{\nu \text{ temperature}}$

$$\rho = \frac{E_{grav}(1-f_B)}{V} = \left[(2s+1)\, N_f\, \frac{7}{8} \right] a T^4$$

$$V = \frac{4}{3}\pi (15\,\text{km})^3$$

solution $T = 1 \sim 10$ MeV

$[a = 7.6 \times 10^{-16}\,\text{J}\,\text{m}^{-3}\,\text{K}^{-4}]$

Supernova neutrino observation in water Č detector

$$\bar{\nu}_e + p \rightarrow n + e^+$$

$e^+ \rightarrow$ Cherenkov light in H_2O

$\bar{\nu}_e$: released when ν-sphere becomes transparent

p: free hydrogen in $H_2O = \frac{2}{18}$ because $\bar{\nu}_e O$ cross section is small

$$\sigma_{\bar{\nu}_e p} = 7.5 \times 10^{-44} \left(\frac{E}{1\,\mathrm{MeV}}\right)^2 \mathrm{cm}^2$$

$$N_{events} = \left[\frac{(1-f_B)\,E_{grav}}{2\,N_{flavors}\,\langle E_\nu \rangle}\right]\left[\frac{\sigma}{4\pi d^2}\right]\left[\frac{2}{18}\,\frac{m_{detector}}{m_N}\right]$$

2: $\bar{\nu}_e$ only

$N_{flavors}$: 3 generations

d: 55 kpc distance to supernova

$$= 5.2\left(\frac{T}{4\,\mathrm{MeV}}\right)\left(\frac{E_{grav}}{2\times 10^{53}\,\mathrm{ergs}}\right)\left(\frac{1-f_B}{0.9}\right)\left(\frac{1}{N_{flavors}/3}\right) *$$

$$* \left(\frac{m_{detector}}{1\,\mathrm{kton}}\right)\left(\frac{55\,\mathrm{kpc}}{d}\right)^2$$

Experiment	$m_{detector}$ (kTon)	N_{events} predicted
Kamiokande	2,14	11
IMB	5	13*
Mont Blanc	0,09	0.6**

* observed 6 which is OK after correcting for threshold

** observed 5 at a time 4.8 hours earlier: statistical fluctuation?

also the Baksan experiment observed a marginal signal.

Spectacular particle physics with < 20 events

- N_ν flavors < 7
- $\gamma T_\nu > 1.6 \times 10^5$ years
- $m_\nu \lesssim 20$ eV

$$\Delta t = \frac{d}{v_1} - \frac{d}{v_2} = d\left(\frac{v_2 - v_1}{v_1 v_2}\right) \simeq \frac{d}{c^2}(v_2 - v_1)$$

$$= \frac{d}{c}\left(\frac{v_2}{c} - \frac{v_1}{c}\right) = \frac{d}{c}\left[\left(1 + \frac{v_2}{c}\right) - \left(1 + \frac{v_1}{c}\right)\right]$$

$$\cong \frac{d}{c}\left[\frac{1}{2\gamma_2^2} - \frac{1}{2\gamma_1^2}\right]$$

$$\boxed{\Delta t = \frac{d}{2c}\left(\frac{1}{E_2^2} - \frac{1}{E_1^2}\right) m_\nu^2 \quad \lesssim 20\,\text{eV}}$$

using $\gamma = \frac{E}{m} = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \Rightarrow \quad \frac{v}{c} = 1 - \frac{1}{2\gamma^2}$

numerical result for $E_1 = 35$ MeV, $E_2 = 7.5$ MeV and $\Delta t = 13$ seconds.

Summary: for historic 1987A observation, see

>> http://sn1987a-20th.physics.uci.edu/Program1.htm

$$\#\ \text{neutrinos} = \frac{1.4\, M_\odot c^2}{10\,\text{MeV}} = 10^{59}$$

$2 \cdot 10^{30}$ kg

$2 \cdot 10^{-29}$ kg

detection for 1 proton

$$\frac{\sigma(\nu + p)}{4\pi d^2} = \frac{10^{-43}\,\text{cm}^2}{4\pi\,(8 \cdot 10^{21}\,\text{cm})^2} = 10^{-88}$$

$$\frac{1\ \text{kTon detector}}{m_p} = \frac{10^6\,\text{kg}}{10^{-27}\,\text{kg}} = 10^{33}\ \text{protons}$$

$$\#\ \nu'\ \text{detected} = 10^{-88} \cdot 10^{33} \cdot 10^{59} = 10^4$$

per kTon.

[from 8 kpc

for LMC $10^4 \times \frac{2}{18} \times \left(\frac{8}{55}\right)^2 = 23$ per kiloton ✓

and SuperK factor 2 too large

Cosmic rays

flux: • solar (bursts, accelerated at termination shock or anomalous cosmic rays) ~ 11 year cycle
• galactic (most)
• extra-galactic

(also electrons)

acceleration on the back-of-the-envelope: Hillas formula

$$R_{acc} > R_{gyro} = \frac{E}{ZeBv}$$

the gyroradius of the particle must be contained in the size of the accelerating region R_{acc} over which the field B extends

$$\boxed{E < (Ze)BR_{acc}v \simeq (Ze)BR_{acc}c}$$

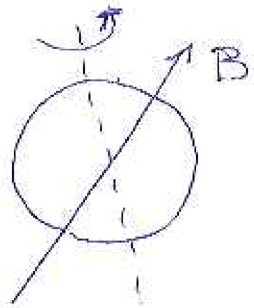
or $E = \eta_{eff}\,(Ze)\,BR_{acc}v$ (dimensional analysis)

η_{eff} ↓ an efficiency factor ($0.01 \sim 0.1$ for shocks)

⇝ Hillas plot

⇝ emf of a pulsar: replace $v \to \omega = \frac{2\pi R_{acc}}{T}$

$$E = \eta_{eff}\,(Ze)\,2\,\underbrace{B\,\frac{\pi R_{acc}^2}{T}}$$



Faraday $\quad \dfrac{d\phi}{dt} = B\,\dfrac{\pi R_{acc}^2}{T}$

rate at which the flux of the spinning pulsar changes

$$\left.\begin{array}{l} B = 10^{12}\ \text{Gauss} \\ R_{acc} = 10\ \text{km} \\ T = 10^{-3}\ \text{s} \end{array}\right\} \rightarrow 10^{18}\ \text{eV}$$

(higher for magnetars)
$B = 10^{15}$ Gauss

composition

note : • even-odd differences in binding energy

• Li, Be, B more abundant : e.g. boron is produced by oxygen interacting with galactic plane (1 proton cm^{-3})
$\rightarrow$ they are spallation products

$$\frac{dN}{dx} = -\frac{N}{\lambda} \qquad N = e^{-\frac{x}{\lambda}}$$

$$\lambda = \frac{1}{n\sigma}$$

in cosmic ray physics one discusses propagation through column densities, e.g. air $10^3 \frac{g}{cm^2}$

$$x(cm) \longrightarrow x\left(\frac{g}{cm^2}\right)$$

$$\lambda(cm) \longrightarrow \frac{1}{\underbrace{\left(\frac{N_A}{A}\right)}_{\text{number of nuclei per gram of matter}}\sigma}$$

$e^{-\frac{x}{\lambda}}$ how many interact in a distance x

example: λ_{p-air}

$A = 14.4$ (average O, N)
$\sigma = 300\,mb = 3 \times 10^{-25}\,cm^2$
$N_A = 6 \times 10^{23}$

$$\lambda_{p-air} = 80 \frac{g}{cm^2}$$

atmosphere $\simeq$ 10 interaction lengths.

- back to spallation: given the $\left\{\begin{array}{l}\text{cosmic ray flux}\\ \text{cross sections}\end{array}\right.$ it takes $5 \frac{g}{cm^2}$ in order to produce the spallation products. This is a travel distance of $\left[\frac{5g}{cm^2}\right] / \left[\frac{m_p}{cm^3}\right] \simeq 3 \times 10^{24}\,cm = 3\,Mpc$

$$m_p = 1.7 \times 10^{-24}\,g$$

note that the hydrogen density in the Galactic plane is

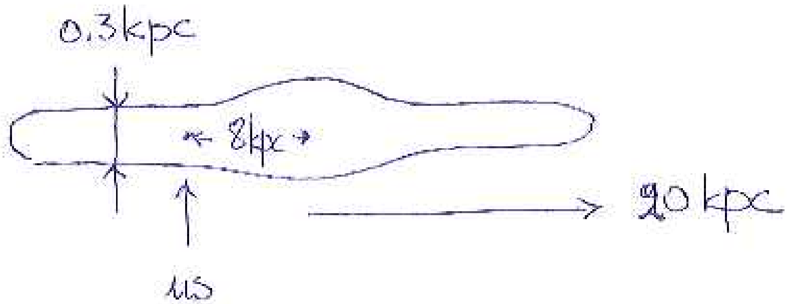
$$n_H \simeq \frac{10^9 M_\odot \cdot N_A}{V_{Gal}} \simeq 1\ \text{cm}^{-3} \qquad \simeq (0.01) 10^{11} \text{stars}$$

($M_\odot \sim 10^{33}$ g)

$$V_{Gal} \simeq 10^{66} \sim 10^{67}\ \text{cm}^3$$

- back to spallation:
  cosmic rays diffuse through the interstellar medium for $\frac{1\,\text{Mpc}}{c} \simeq 10^6$ years (leaky box). This time

  is energy (gyroradius ~ E in 3 μG field) dependent.



- Interstellar Medium (ISM) in galactic disk

per cm^3

cosmic rays $\quad 10^{-12} \frac{\text{erg}}{\text{cm}^3}$ $\left(\text{later} \int^{4\pi} \frac{1}{c}\frac{dN}{dE} E\right)$ $\qquad$ 0.5 eV

magnetic energy $\frac{B^2}{2\mu_0} = \frac{(3\times 10^{-10})^2}{8\pi \cdot 10^{-7}} \frac{\text{J}}{\text{m}^3}$ $\qquad$ 1 eV

background γ' $\left(n_\gamma kT = \frac{400}{\text{cm}^3}\ 8.6\times 10^{-5} \frac{\text{eV}}{°\text{K}} \cdot 2.7\right) \cdot 3.15$ $\qquad$ 0.25 eV

light in stars $\qquad$ 0.6 eV

- energy of particle matching scale height

$$R_{gyro} = 0.3\,\mathrm{kpc} = \frac{E}{Zec\,(3\mu G)}$$

$$E = 10^{18}\,\mathrm{eV}$$

particles can be contained for $E < 10^{18}\,\mathrm{eV}$ (knee?)
depends on Z, high Z contained at higher energies

- do galactic cosmic rays originate in supernova remnants?
the energetics argument

measured cosmic ray flux (called F, ϕ or $\frac{dN}{dE}$) in units of particles / [GeV cm² sr]. From flux $= \rho_E \times v$

$$\text{energy density} \equiv \rho_E = \frac{4\pi}{c} \int E \frac{dN}{dE}\, dE = 0.5\,\frac{\mathrm{eV}}{\mathrm{cm}^3} = 10^{-12}\,\frac{\mathrm{erg}}{\mathrm{cm}^3}$$

accelerating power needed to replace the particles every 3×10^6 yr is

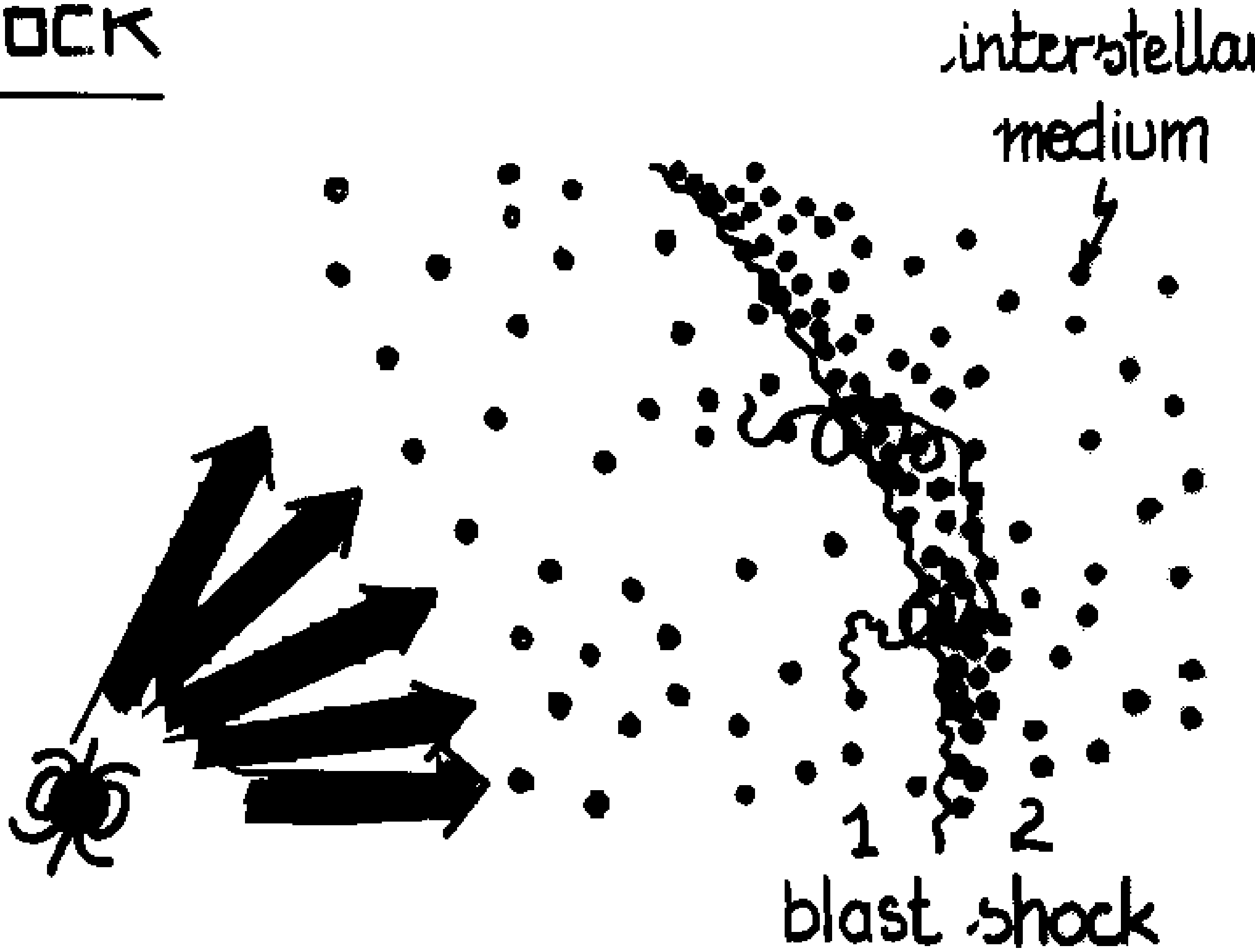
$$\frac{\rho_E}{\tau_{life}} \simeq 10^{-26}\,\frac{\mathrm{erg}}{\mathrm{cm}^3\,\mathrm{s}} \text{ or } 10^{41}\,\frac{\mathrm{erg}}{\mathrm{s}} \text{ in } V_{gal} = 10^{67}\,\mathrm{cm}^3$$

this corresponds to 1 supernova every 30 years or

$$\left[\frac{\dot{M}_\odot}{(1-f_B)} \cdot (0.03)\right] \Big/ \left[30\,\text{years}\right] \cong 10^{41}\,\frac{\mathrm{erg}}{\mathrm{s}} \quad \text{match!}$$

↳ acceleration efficiency ~~0.1~~ 0.01

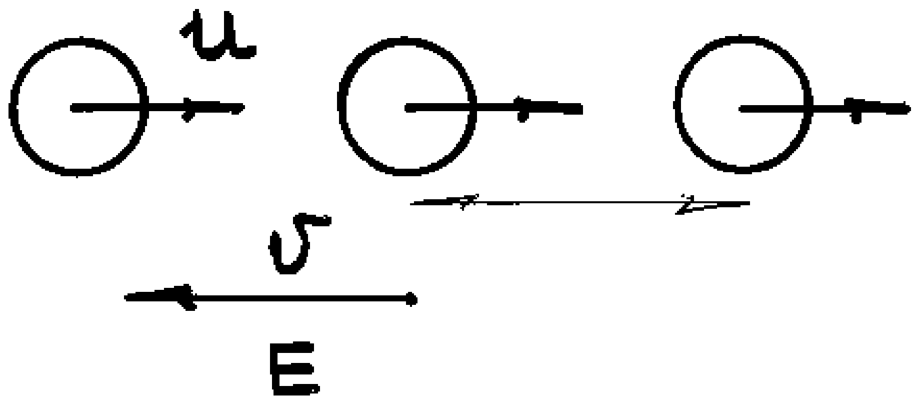
SHOCK



$$v_2 > v_1$$

- $$\frac{\Delta E}{E} = \frac{v_2 - v_1}{c}$$

FERMI 49 : particle collides with stellar clouds



1pc ~ 1ly

$$\frac{\Delta E}{E} \sim \frac{u}{v}\left(\frac{u}{c}+\frac{v}{c}\right) \qquad u \ll v \simeq c$$

$$\boxed{\frac{\Delta E}{E} \sim \frac{u}{c} + \left(\frac{u}{c}\right)^2}$$

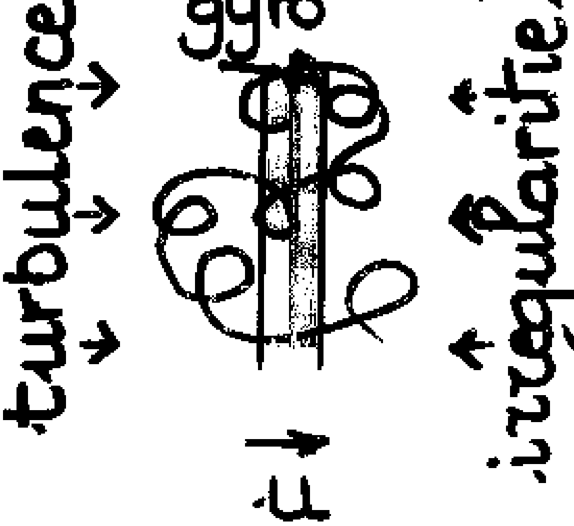
- some clouds ($\frac{v-u}{v+u}$ fraction, Doppler come from the wrong direction

$$\frac{\Delta E}{E} \sim \frac{u}{v}\left(\frac{u}{c}+\frac{v}{c}\right)\left(\frac{u+v}{v}\right) - \frac{u}{v}\left(-\frac{u}{c}+\frac{v}{c}\right)\left(\frac{v-u}{v}\right.$$

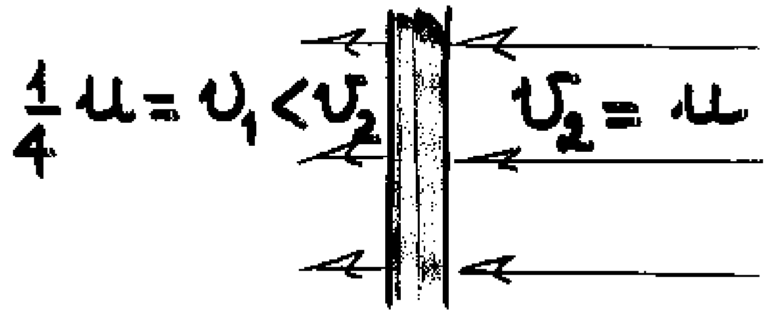
- u/c term cancels

FERMI 53 : only good encounters

- $R_{gyro} < R$



- shock at rest

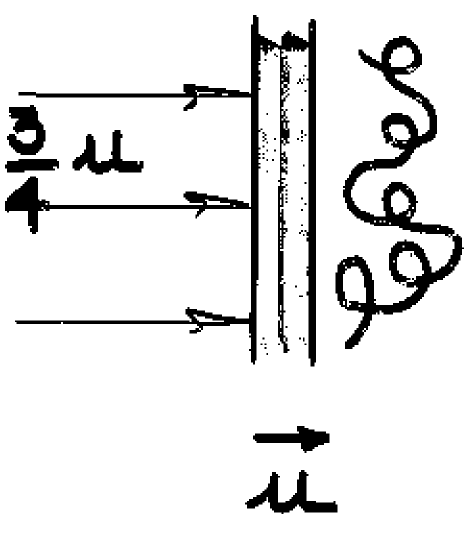


o

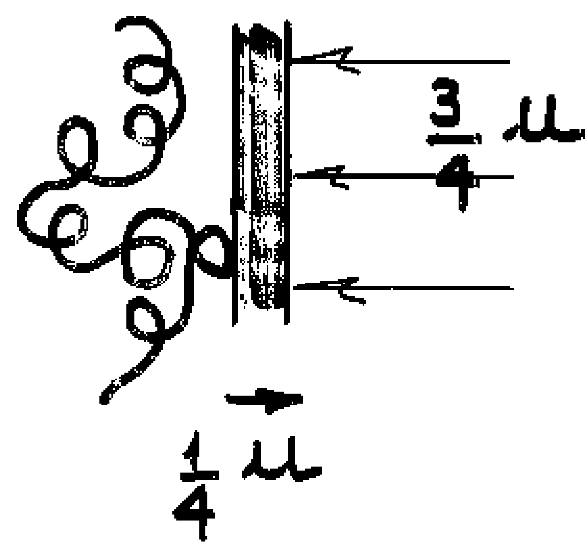
$$v_1 = f\, v_2 \simeq f u$$

$$f \simeq 1/4$$

- frame of upstream particle



- frame of downstream particle



examples: earth's magnetosphere, solar flares, solar wind, ..., water running in flat sink ...

First order Fermi $\rightsquigarrow E^{-n}$

after k collisions

- $E = \beta^k E_0$ — β energygain in single encounter $\frac{\Delta E}{E_0} = \frac{u}{c}$
- $N = P^k N_0$ — P probability to cross shock (again)

$$\boxed{\frac{dN}{dE} \sim E^{-1 + \frac{\ln P}{\ln \beta}}}$$

−1 for $f = \frac{1}{4}$

cosmic rays

$$\frac{dN}{dE} \sim E^{-2.7} = E^{-2.1-0.6}$$

low energy secondary particles

SHOCKS (ctd): maximum energy limited by acceleration time

shock velocity ~ 0.1

- $$\frac{\Delta E}{\Delta t} = K \left(\frac{v}{c}\right)^2 ZeBc$$

efficiency ~ 0.1

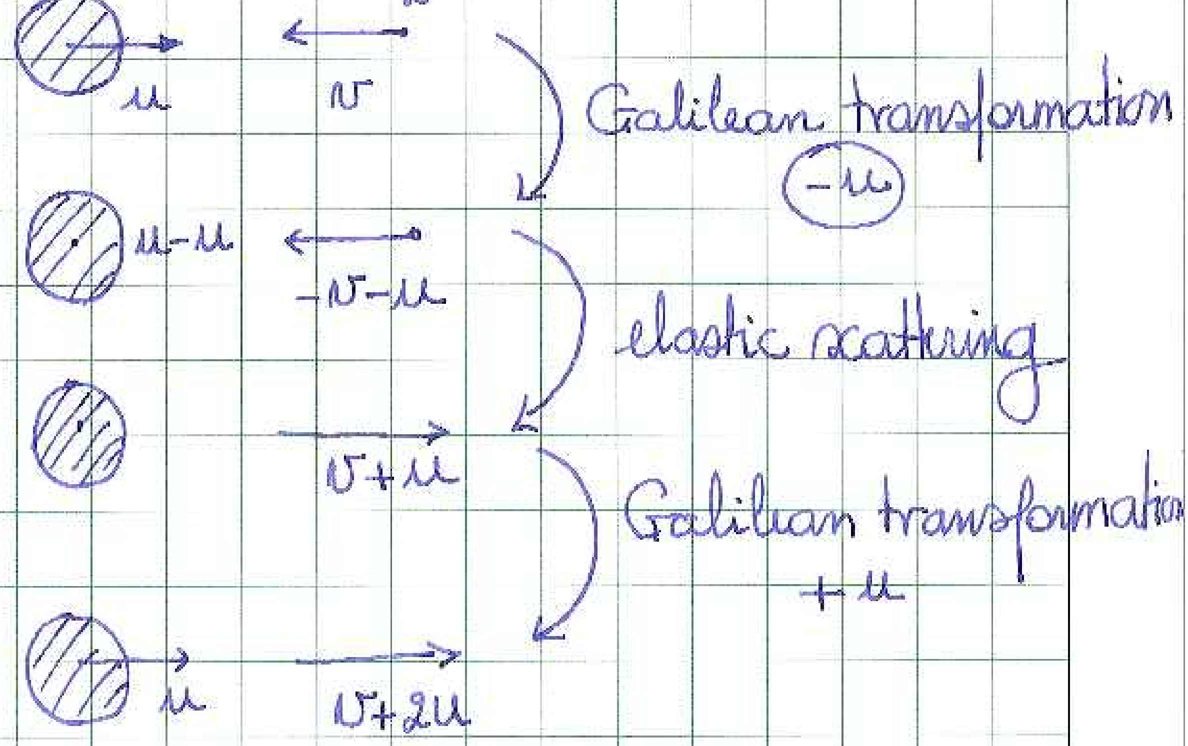
$$v \Delta t = \ell$$

size of shock

- $$E = K \left(\frac{v}{c}\right) ZeB\ell$$

$$\boxed{E < ZeB\ell}$$

$$E = \frac{1}{2} m v^2$$



$$\frac{\Delta E}{E} = \frac{(v+2u)^2 - v^2}{v^2} = 4\frac{u}{v}\left(1+\frac{u}{v}\right)$$

$$\frac{\Delta E}{E} \sim \frac{u}{c} + \frac{u^2}{c^2} \quad (\text{relativistically } u \ll v \simeq c)$$

loss $u \to -u$

$$\left(\frac{\Delta E}{E}\right)_{net} = 4\frac{u}{v}\left(1+\frac{u}{v}\right)\frac{1}{2}\left(\frac{u+v}{v}\right) \underbrace{- 4\frac{u}{v}\left(1-\frac{u}{v}\right)\frac{1}{2}\left(\frac{-u+v}{v}\right)}$$

$$= 8\left(\frac{u}{v}\right)^2$$

$$\frac{dN}{dE} = e^{+\frac{\ln P}{\ln \beta} - 1}$$

geometry

$$P \sim \frac{v_1}{c}$$

$$\beta \sim \frac{v_1 - v_2}{c}$$

$$\boxed{\frac{\ln P}{\ln \beta} = 3\,\frac{v_1}{v_1 - v_2}}$$

$$\frac{\ln P}{\ln \beta} = \frac{3}{1 - \frac{v_2}{v_1}} = -1$$

$= 4$ for ionized gas